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SECOND-ORDER FACTORS

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Second-order factors are defined and illustrated in terms of a literal notation, a physical example, a diagrammatic representation, a geometrical example, and the matrix equations relating the first-order and second-order domains. Both kinds of factors are discussed as parameters which may be not only descriptive of the individual objects in a statistical population but also descriptive of the restrictive conditions under which the objects were generated or selected. Second-order factors may be of significance in reconciling the several theories of intelligence. This paper is concerned with test configurations that show simple structure. If such a structure is not revealed, then the second-order domain is indeterminate.

1. *First-Order and Second-Order Factors*

Most of the work that has been done so far in the development of factorial theory has been concerned with the factors obtained from test correlations with or without rotation of axes for the selection of a suitable reference frame. *Factors that are obtained from the test correlation will be called first-order factors* whether they are selected so as to be orthogonal or oblique. We shall now consider the factors that may be determined from the correlations of the first-order factors. *Factors that are obtained from the correlations of the first-order factors will be called second-order factors.* Factors of this type seem to be of fundamental significance in the interpretation of correlated variables.*

Analysis of second-order factors and their relations to those of first-order can be presented in several different ways. We shall describe these two factorial domains in terms of a literal notation, a physical example, a diagrammatic representation, a geometrical example, and the matrix equations relating the two domains.

Consider first a reduced correlation matrix for the tests whose rank is, say, five. The factoring of this correlation matrix determines five arbitrary orthogonal unit reference vectors which may be denoted *I, II, III, IV,* and *V*. This orthogonal reference frame is arbitrary in

* This study of second-order factors is one of a series of investigations in the development of multiple factor analysis and applications to the study of primary mental abilities. This investigation has been supported by a research grant from the Carnegie Corporation of New York. The Psychometric Laboratory has also had support from the Social Science Research Committee of The University of Chicago.

the sense that it is defined by the method of factoring which happens to be used. This reference frame will be regarded as fixed and all other vectors will be defined in terms of this fixed orthogonal frame, which is designated by the subscript m. Let it be assumed that a complete simple structure can be found in the test configuration and let the corresponding primary vectors be denoted A, B, C, D, and E. (These are ordinarily denoted T_A, T_B, etc.) We shall assume that these primary traits are correlated in the experimental population. Then the primary vectors in the test configuration will be separated by acute angles whose cosines are the correlations between the primary traits in the particular group of subjects studied. Let these correlations be listed in a new correlation matrix of order 5×5 showing the correlations between primary factors. This correlation matrix defines the second-order domain just as the correlation matrix for the tests defines the first-order domain.

The simplest case is that in which the five primary factors are uncorrelated, in which case their correlation matrix is a unit matrix so that an analysis of a second-order domain is not immediately indicated. Next would be the case in which the reduced correlation matrix for the primary factors A, B, C, D, and E is of unit rank. There are two types of interpretation for such a situation. The correlations between the primary factors in a particular experimental population may be due to conditions of selection of the subjects, and in this case the correlations would be of no more theoretical importance than the conditions of selection of the subjects. If, on the other hand, the five primary functions A, B, C, D, and E actually do have some parameter in common, then one would expect their intercorrelations to be of unit rank for different experimental groups of subjects that are selected in different ways. In other words, the mere fact that a set of variables, or a set of factors, are correlated does not imply any scientific obligation to find "the" factors that account for the correlations because the factors, if found, might turn out to be as incidental in significance as the conditions by which the subjects happened to be selected. On the other hand, the fact that correlations between variables, or between factors, can be caused by scientifically trivial circumstances does not guarantee that all correlations between variables are of trivial significance. If the correlations between the five primary factors in the present example should turn out to be of unit rank, then this circumstance merits a closer look because such a simplification would not often happen by chance. If the correlations between the primary factors should turn out to be of unit rank for several different experimental groups, then we should have an obligation to ascertain the cause which must transcend the selective conditions.

In order to avoid misunderstanding, perhaps it should be remarked that in factor analysis we are using the term *parameter* in its universal meaning in science. *A parameter is one of the measurements that are used for describing or defining an object or event.* In statistical theory the term *parameter* is frequently used in a more restricted sense as descriptive of the universe as contrasted with a *statistic* which is the corresponding measurement on a sample. We are not using the term in this restricted sense.

Let it be assumed that the five primary factors do have a parameter p in common. Then the five primaries could be expressed in the form

$$\begin{aligned} A &= f(p, a), \\ B &= f(p, b), \\ C &= f(p, c), \\ D &= f(p, d), \\ E &= f(p, e), \end{aligned}$$

where each primary function is defined in terms of a parameter such as a, b, c, d, or e, which is unique to itself and also in terms of another parameter p, which it shares with the functions that define the other primaries. If there should happen to be conspicuous correlation between the parameters a, b, c, d, and e in the particular group of subjects, then the unit rank of the second-order domain would be disturbed. If the correlations of the primaries show unit rank, then, in addition to the parameters a, b, c, d, and e, a second-order parameter or factor p can be postulated.

It should be noted that we now have six parameters, namely, a, b, c, d, e, and p, and since the rank of the test correlations is five, it follows that these six parameters are linearly dependent. In fact, the parameter p is now a linear combination of the other five parameters. We can express these relations by a set of parameters such as A, B, C, D, E, and p, in which p is a linear combination of the five primary parameters. The five primaries are parameters descriptive of the first-order domain, and the parameter or factor p is descriptive of the second-order domain, which is here of unit rank. The second-order parameter is a linear combination of the five primaries that are defined by the original test correlations. If some degree of consistency can be found for these parameters for different groups of subjects, then all of these parameters should represent some aspects of the underlying physical and mental functions.

Consider next a set of correlated primaries A, B, C, D, and E in which the parameter p appears in the first order as in the following example:

$$
\begin{aligned}
A &= f(p\,,a), \\
B &= f(p\,,b), \\
C &= f(p\,,c), \\
D &= f(p\,,d), \\
E &= f(p).
\end{aligned}
$$

The rank of the reduced correlation matrix of the tests would now be five. The five primaries listed above would be correlated and of unit rank. The second-order factor p would be determined from the correlations of the primaries. In this case the communality for the primary factor E would be near unity, thus showing that its total variance is common to the second-order factor p, hence the primary factor E of the first order and the factor p of the second order would be identical. The presence, or absence, of the primary E could be determined by including, or excluding, a few tests in the battery. We see, therefore, that the appearance of a factor in the first order or in the second order may depend on the battery of measurements taken; hence a factor should not be considered as intrinsically different because it appears in the second order. This circumstance can be determined by the selection of the test battery.

On the other hand, a parameter which always appears in the measurements in association with some other function would not appear as in the primary E and it would be discovered experimentally nearly always in the second order. Such a limitation could be introduced by the physical nature of the attribute which the factor represents, so that in such a case the second order would represent something fundamentally different from that of the first order. A single factor study is not likely to reveal whether a second order parameter is fundamentally different from the parameters of first order or whether the differentiation is caused merely by the selection of the test battery.

In the following example we have another combination of primaries,

$$
\begin{aligned}
A &= f(p\,,a), \\
B &= f(p\,,b), \\
C &= f(p\,,c), \\
D &= f(p\,,d), \\
E &= f(e).
\end{aligned}
$$

In this example the reduced rank of the test correlations would again be five. The correlations of the primaries would show unit rank for $A, B, C,$ and D. The factor E would be orthogonal to the rest of the system so that its row and column would have side correlations of zero. The correlations of primaries would not be of unit rank if we consider the whole table of order 5×5 but it would be of unit rank if

we consider only the 4×4 table for *A, B, C,* and *D.* Relations of this kind can be found by inspection of the correlations of the primaries and they may be indicative of the underlying order in the domain that is being investigated.

The principles of a second-order domain have been discussed here in terms of the simple case in which that domain is of unit rank so that there is only one general second-order factor. It should be evident that the organization of the second-order parameters can be of any rank and complexity. For example, the rank may be higher than one, and the second-order factors may extend to all of the primaries or only to some of them. The possibility of third-order and higher-order factors must be recognized but their experimental identification is of increasing difficulty the higher the order because of the instability of such a superstructure on practically feasible experimental data. The number of second-order factors that can be determined from a given number of linearly independent primary factors follows the same restrictive relations that govern the number of primary factors that can be determined from a given number of tests. Thus, for example, it is not to be expected that three second-order factors will be determinate from only five primary factors for the same reason that three primary factors cannot be determined from five tests. Furthermore, it is entirely possible in the same data for the first-order domain to give clear interpretation of a set of primary factors and for the second-order domain to be indeterminate or ambiguous.

2. *The Box Example*

In order to illustrate the nature of first- and second-order factors, we shall make use of populations of simple objects or geometrical figures and their measurable properties instead of dealing with these factors merely as logical abstractions. We have used a population of rectangular boxes and their measurable attributes to illustrate the principles of correlated primary factors and we can use them also for the present discussion.*

A random collection of rectangular boxes was represented by the three measurements length (x), width (y), and height (z). A list of measurements was prepared which could be made on each box, such as the diagonal of the front face, the area of the top surface, the length of a vertical edge, and so on. Each of these measurements represented a test score and each box represented an individual member of the statistical population. The correlations between the measurements were computed and analyzed factorially as if we did not know

*Thurstone, L. L. Current issues in factor analysis. *Psychol. Bull.*, 1940, 37, p. 222.

anything about the exact nature of each measurement, which was treated as a test score of unknown factorial composition. As has been shown previously, the analysis revealed three factors in the correlations for the particular set of measurements used. The configuration showed a complete simple structure and a set of primary vectors was determined by the configuration. These three primaries represented the three basic parameters in terms of which all the test measurements had been expressed.

The three primary vectors were separated by acute angles whose cosines represented the correlations between the three basic parameters that were used in setting up the box example. These three correlations could be assembled into a small correlation matrix of order 3×3. The physical interpretation of the positive correlations was that large boxes tend to have all of their dimensions larger than small boxes. In other words, if one of the dimensions of a box shape is, say, six feet, the other dimensions of the box are not likely to be of the order of, say, two or three inches. The table of correlations of the three primary factors X, Y, Z, could be represented by a single common factor. This factor would be a second-order factor. It would, no doubt, be interpreted as a size factor in the box example. If this second-order size factor were denoted s, we should have four parameters for describing the box shapes, namely, the three dimensions, x, y, and z and the size factor s. These four parameters or factors would be linearly dependent because the rank of the correlation matrix of the tests was three.

In the case of the box example, a size factor or parameter could be determined in the first-order if desired. For this purpose we could use the first centroid axis, the major principal axis, or the volume vector, all of which can be easily defined in the first-order system of test vectors. The four parameters so chosen would also be linearly dependent. If we wanted to use only three linearly independent parameters including a size factor, that could be done in the first order by choosing, say, the two ratios $x/y = r_1$ and $x/z = r_2$ as well as the volume vector v. These three factors would be linearly independent but they would be correlated. The latitude with which we can choose simplifying parameters for the box example is determined in part by the fact that three factors can nearly always be represented by a common factor whereas this is not the case when the rank is higher than three.

3. *Diagrammatic Representation*

The relations between the first-order and the second-order domains can be represented diagrammatically as shown in ***Figures 1,***

2, and *3.* In *Figure 1* we have a set of eight tests whose correlations are accounted for by five primary factors, *A, B, C, D,* and *E,* which are uncorrelated. The factor *A,* for example, is present in the common factor variances of tests 1, 2, and 4. The primary factor *E* is present in the common factor variances of all the tests, and hence *E* would be called a general factor for the particular battery. Since it is orthogonal to all the other primary factors it may be called an *orthogonal general factor of the first order.* In order to determine the nature of the factor *E* it would be necessary to study it in different test batteries so that one could predict with certainty when the factor would be present and when it would be absent from a test. Since the primary factors are here represented as uncorrelated, the matrix of correlations of the primary factors would be an identity matrix and there would be no immediate provocation to investigate a second-order domain.

In *Figure 2* we have represented a set of tests and five primary factors *A, B, C, D,* and *E.* (We are not here concerned as to whether the particular number of tests represented in this diagram is adequate for the determination of five primary factors. The purpose of these diagrams is merely to show the nature of the relations between the two domains.) The rank of the correlation matrix of the tests would here be five, which corresponds to the number of linearly independent primary factors. In the present case we should find that the primary factors are themselves correlated. The matrix of correlations of these primaries would be of order 5×5 and it would be of unit rank. The correlations between the primary factors could therefore be accounted for by a single general second-order factor that is denoted *G.* If both the first-order and second-order factors were to be used for the description of the tests and the relations, we should have six parameters which would be linearly dependent because the rank of the correlations of the tests is only five. In fact, the saturation of each test with the second-order factor *G* would be a linear combination of the saturations of the test with the five primaries of the first order. None of the primary factors are general factors in this figure.

In *Figure 3* we have a more complex relation in that the correlation matrix for the primary factors would be of rank two. One of the second-order factors is here shown to be common to all but one of the primary factors, one of the second-order factors is a factorial doublet in that it represents additional correlation between the primaries *B* and *D,* and the primary factor *A* is orthogonal to the rest of the primaries so that it does not participate in the second-order domain. This diagram is drawn merely to illustrate the variations in complex-

ity that may be found in factorial studies.

The two types of general factor here shown in *Figures 1* and *2* have some interesting differences. The general factor E of *Figure 1* is independent of the other primary factors while the general factor G in *Figure 2* is present in all of the other factors. Hence we must conclude that a second-order general factor is a part of, and must participate in, the definition of the other factors while the orthogonal general factor E of *Figure 1* is, by definition, independent of the other primary factors. It is evident, therefore, that a general second-order factor is likely to be of more fundamental significance for the domain in question than a general orthogonal first-order factor. An orthogonal general factor of the first order might operate in a test without any group factor whereas a second-order general factor would operate, ordinarily, through the mechanism of some function that could be identified as a group factor, a primary factor, or a special ability.

The factor patterns corresponding to the relations shown diagrammatically in these figures are given in *Tables 1, 2* and *3*. *Table 1* shows the factor pattern for *Figure 1*. Here the orthogonal general factor E is identified by the fact that all entries of its column are filled. *Table 2* shows the factor pattern for *Figure 2*. Here it is seen by the factor pattern that a group such as tests 1, 4, 5, and 7 have no primary factor in common and that hence their correlations would be determined only by the second-order general factor G. The determinant of the correlations for these four tests (the tetrad difference) would therefore vanish. The second-order factor matrix is also shown in this table with only one factor G to correspond to this example.

The question might be raised whether both types of general factor could be present in the same battery. That seems possible. In that case a simple structure could define the primary factors A, B, C, and D but not E in the particular battery of *Figure 1*. This factor could be assumed arbitrarily to be orthogonal to the other factors, but then the line GE of *Figure 2* would be erased to correspond to the fact that E is orthogonal to the other factors. One or more second-order general factors could be found in the correlated primaries. If the correlations of A, B, C, and D were of unit rank, another alternative would be to set E in such a relation to the other primary vectors as to maintain the unit rank with the second-order general factor. It might then be found that the vector E has non-vanishing projections on all the test vectors, in which case both types of general factor would be assumed to be a possible set of explanatory parameters for the battery in question. It must be remembered that these various locations of the reference frame for the explanatory parameters in both the first-order and the second-order domains have validity only in so far

as they are suggestive of fruitful scientific interpretation. If this is not the purpose, then the factorial resolution might as well remain in the arbitrary orthogonal factors produced by factoring the given test correlations—or, better still, by not doing the factoring at all.

It might be asked how the correlations of a test battery can be resolved into a second-order domain of unit rank which is lower than the rank of the test correlations. The transitions can be regarded geometrically. The unit test vectors usually define a space of as many dimensions as there are tests. When the reduced correlation matrix is considered, its rank is frequently lower than its order. Hence the reduction from the number of tests n to the number of primary factors r represents a reduction from the total variance of the tests to the common factor variance. The complete correlation matrix for the primary factors represents a set of r unit vectors in as many dimensions, the dimensionality of the common factor space. The reduced form of this matrix for the example of *Figure 2* would have unit rank because the side correlations are determined only by that which the primaries have in common, namely, the second-order general factor.

4. *Group Factors and Primary Factors*

In *Figure 4* we have a diagrammatic representation of a different kind of resolution of factors in the second-order domain and their relation to the primary factors. In this example the rank of the correlation matrix of tests is assumed to be five as represented by as many primary factors $A, B, C, D,$ and E. Let it be assumed that the correlation matrix for these five primaries is of unit rank. The general second-order factor G then accounts for the observed correlations of the primary factors. If the five linearly independent primary unit vectors and the second-order unit vector G are to be represented in the same space, the dimensionality of this space must be six. It is possible to locate in this augmented space another set of unit vectors $a, b, c, d,$ and e which are mutually orthogonal and which are also orthogonal to the unit vector G. Then we have the orthogonal reference frame $G, a, b, c, d,$ and e which defines the six dimensions of the first- and second-order factors but not the test space. The five linearly independent primary factors define a five-dimensional space corresponding to the rank of the test correlations, and this space is a part of the total six-dimensional space of this representation.

The unit vector a is a linear combination of the unit vector G and the primary vector A. The relation is similar for the other primary vectors. The primary vectors $A, B, C, D,$ and E are correlated and of unit rank whereas the vectors, $a, b, c, d,$ and e are arbitrarily

set orthogonal to each other. In general, if the rank of the test correlations is r, and if the correlations of the primaries are of unit rank, then the primaries define a unit vector G for a general second-order factor in an augmented space of dimensionality $(r + 1)$ and also a set of r mutually orthogonal unit vectors each of which is in the plane of the second-order general factor and one of the primaries. These vectors are arbitrarily set orthogonal to the general second-order factor and they are called *group factors*. In *Figure 4* the primary factors are denoted A, B, C, D, and E and the group factors are denoted a, b, c, d, and e. With this resolution we have $(r + 1)$ linearly dependent factors which represent the test correlations of rank r. This type of resolution is preferred by some students who use the reference frame G, a, b, c, d, and e because it is orthogonal rather than the frame G, A, B, C, D, and E which is oblique.

5. *A General Second-Order Factor*

The algebraic and computational relations between the first-order and the second-order domains will be shown for the case of a single general second-order factor because of the interest of this case for the psychological controversies of the past forty years about Spearman's general intellective factor. The algebraic and computational relations to be shown can be generalized to second-order domains of higher than unit rank. It must be remembered, however, that the restriction of our discussion to unit rank for the second-order domain does not in any way imply that such low rank is always to be expected. The methods of analysis can be readily extended to a second order of higher rank when the data indicate a determinate second-order configuration. In any case, the second-order rank should be considerably lower than the rank r of the first-order factors in order to justify interpretation.

The primary vectors constitute a set of r linearly independent unit vectors that define a space of dimensionality equal to the rank of the test correlations. In order to represent a general second-order factor as a unit vector in the same configuration it is necessary to augment the dimensionality to $(r + 1)$ dimensions. A second-order domain of rank two would thus require an augmented space of dimensionality $(r + 2)$. The projections of the test vectors on these additional vectors in the augmented space can, however, be expressed as linear combinations of the test projections on the primary vectors or on any set of r linearly independent vectors in the common factor space. The procedures for determining these saturations will be shown without writing explicitly the $(r + 1)$ co-ordinates of the second-order unit vectors in the augmented space.

The present discussion is confined to factorial data that satisfy two conditions, namely 1) that a complete simple structure is revealed in the test configuration and 2) that the second-order correlation matrix is of unit rank. These methods can be adapted to the analysis of less than r primary factors and the methods can be adapted to higher second-order rank.

One of two objectives will be assumed, namely (1) to determine the projections (saturations) of the tests on the second-order factor in addition to the projections on the primary reference vectors or (2) to determine the projections on the second-order factor and also on the orthogonal group factors. It will be convenient to discuss the algebraic relations under four cases because of the different computational routes that may be chosen. These four cases are:

Case 1. Transformation from **F** *to* **V** *including the column vector* **G**

This transformation is shown in rectangular notation in *Table 4* for the equation

$$\boldsymbol{F}_{jm}\,\boldsymbol{\Psi}_{mp} = \boldsymbol{V}_{jp}\,, \tag{1}$$

in which the matrix $\boldsymbol{V}_{jp}$ has an extra column for the second-order factor G with elements v_{jg}, which may also be denoted r_{jg} because these are the correlations between the tests j and the general factor G. The transformation matrix $\boldsymbol{\Psi}_{mp}$ is identical with Λ_{mp} except for the added column G with elements ψ_{mg} which are to be determined. Consider the matrix $\boldsymbol{T}$ as an extension of the factor matrix $\boldsymbol{F}$. The rows of $\boldsymbol{T}$ give the direction cosines of the primary vectors T_t with elements t_{tm}. The same transformation gives

$$\boldsymbol{T}_{tm}\,\boldsymbol{\Psi}_{mp} = \boldsymbol{V}_{tp}\,, \tag{2}$$

which is the diagonal matrix $\boldsymbol{D}$ except for the first column. Applying the transformation $\boldsymbol{\Psi}_{mg}$ we have

$$\boldsymbol{T}_{tm}\,\boldsymbol{\Psi}_{mg} = r_{tg}\,, \tag{3}$$

where r_{tg} is the first column of $\boldsymbol{V}_t$ and its elements are the correlations of the primary factors with the general factor G. These are known from the factoring of the unit-rank correlation matrix for the primaries. Then

$$\boldsymbol{\Psi}_{mg} = \boldsymbol{T}^{-1}{}_{tm}\, r_{tg}\,, \tag{4}$$

and, since $\boldsymbol{T} = \boldsymbol{D}\,\Lambda^{-1}$, we have

$$\boldsymbol{\Psi}_{mg} = \Lambda\,\boldsymbol{D}^{-1}\, r_{tg}\,, \tag{5}$$

from which the first column of Ψ_{mg} can be computed. Hence the column G of the augmented oblique factor matrix V becomes known.

Case 2. Transformation from **F** *to* **U** *including group factors and general factor* G

Here the computation starts again with the orthogonal factor matrix F and the objective is to determine the saturations of the tests j with the r group factors and the second-order general factor G. This transformation is also shown in *Table 4* in rectangular notation by the equation

$$F_{jm}\,\Psi_{mw} = U_{jw}\,, \tag{6}$$

where U is the factor matrix showing the projections of tests j on group factors and general factor G. These $(r + 1)$ mutually orthogonal factors will be designated by the subscript w. The first column of this matrix is again the column of correlations r_{jg}. If the same transformation is applied to the matrix T for the primary vectors, we have

$$T_{tm}\,\Psi_{mw} = U_{tw}\,, \tag{7}$$

which is also a diagonal matrix except for the first column which contains the correlations r_{tg} between the primary factors and the second-order general factor G. These saturations can be determined from the unit-rank correlation matrix $TT' = R_t$ for the primary factors.* Consider the first row of U_t. The two entries in this row show the direction cosines of T_A in terms of the orthogonal frame G, a, b, and c. The primary vector T_A is a linear combination of the two orthogonal unit vectors G and a. Hence, when r_{1g} is known, we have

$$r^2_{1g} + u^2_{12} = 1\,, \tag{8}$$

or

$$r^2_{Ag} + u^2_{Aa} = 1\,, \tag{9}$$

so that the element u_{Aa} is known. The other diagonal elements of U_t are determined in the same way so that, for example,

$$r^2_{Bg} + u^2_{Bb} = 1\,. \tag{10}$$

When the matrix U_t is known, we have, by (7),

$$\Psi_{mw} = T^{-1}\,U_t\,, \tag{11}$$

* Elsewhere we have denoted this matrix R_{pq} but we are here using the subscript t for the primary vectors T_t and reserving the subscripts p and q for the primary reference vectors A, B, and C. Hence the correlations of the primary factors are here denoted R_t instead of R_{pq}.

or

$$\Psi_{mw} = \Lambda D^{-1} U_t \tag{12}$$

so that the transformation Ψ_{mw} is known. The saturations of tests j on the second-order general factor G and the group factors w can then be computed.

The transformation matrix Ψ_{mw} represents a rigid rotation from one orthogonal frame to another orthogonal frame, and hence this transformation matrix must be orthogonal by rows. A fourth row could be added to Ψ_{mw} for a fourth orthogonal unit vector IV with cell entries which normalize each column. Then we should have an orthogonal matrix of order 4×4.

Case 3. Transformation from V *to* U *including the group factors and column vector* G.

Here it is assumed that the computations are to be made from the oblique factor matrix V. In *Table 5* we have the transformation equation in rectangular notation, namely,

$$V_{jp} \Psi_{pw} = U_{jw}, \tag{13}$$

which gives the saturations of the tests j on the group factors and on the general factor. If the factor matrix V is extended to include the primary vectors T_t we have the diagonal matrix D. Applying the same transformation to D we have

$$D_{tp} \Psi_{pw} = U_t \tag{14}$$

so that

$$\Psi_{pw} = D^{-1} U_t . \tag{15}$$

When the elements of U_t have been determined as for *Case 2,* the transformation Ψ_{pw} can be written by merely adjusting the rows of U_t by the multipliers of $D^{-1}{}_{tp}$. The transformation Ψ_{pw} is then known.

Case 4. Transformation from V *to column vector* G

This is the simplest case and perhaps the most useful as regards the second-order domain. The matrix V is known in determining the simple structure and the primaries. The saturations of the tests j on the second-order general factor G are of interest and these can be determined as linear combinations of the columns of V. Here we have the transformation shown in *Table 5,* namely,

$$V_{jp} \Psi_{pg} = r_{jg} . \tag{16}$$

Applying the same transformation to D_{tp}, we have

$$D_{tp} \Psi_{pg} = r_{tg} . \tag{17}$$

The elements of the column vector r_{tg} are known from the correlation matrix $\boldsymbol{TT}' = \boldsymbol{R}_t$ of the primaries. Then

$$\Psi_{pg} = \boldsymbol{D}^{-1}{}_{tp}\, r_{tg} , \tag{18}$$

and hence the column vector Ψ_{pg} is known. In computing, it is only necessary to multiply the elements r_{tg} by the corresponding diagonal elements of $\boldsymbol{D}^{-1}{}_{tp}$ to determine Ψ_{pg}. The desired column vector r_{jg} can then be determined.

6. *A Trapezoid Population*

In previous studies of factorial theory it has been found useful to illustrate the principles by means of a population of simple physical objects or geometrical figures. The box population was used to illustrate three correlated factors and their physical interpretation. In the present case we want four factors in the first-order domain which by their correlations of unit rank determine a general second-order factor. The correlations of three variables can nearly always be accounted for by a single factor and hence it seems better to choose a four-dimensional system in which the existence of a second-order general factor is more clearly indicated by the unit rank of the correlations of four primary factors. For the present physical illustration we have chosen a population of trapezoids whose shapes are determined by four primary parameters or factors.

The measurements on the trapezoids are indicated in *Figure 5.* The base line is bisected and the length of each half is denoted by the parameter *c*. An ordinate is erected at this midpoint and its length is *h*. This ordinate divides the top section into two parts which are denoted *a* and *b* as shown. These four parameters, *a, b, c,* and *h,* completely determine the figure. The test battery was represented by sixteen measurements which are drawn in the figure. The parameters *a, b, c,* and *h* are given code numbers 1, 2, 3, and 4, respectively. Variables (12) and (13) are the two areas as shown. The sum of (12) and (13) equals the total area of the trapezoid. In general, each of these measurements is a function of two or three of the parameters but not of all four of them and hence we should expect a simple structure in this set of measurements. There is a rather general impression that a simple structure is necessarily confined to the positive manifold. In order to offset this impression we included here three additional measures which extend the simple structure beyond the positive manifold. The three additional measures are as follows:

$$14 = (1) \;/\; (2) = a/b$$
$$15 = (2) \;/\; (3) = b/c$$
$$16 = (1) \;/\; (3) = a/c$$

These three measures will necessarily introduce negative saturations on some of the basic factors.

In *Table 6* we have a list of dimensions for a set of thirty-two trapezoids. These will constitute the trapzeoid population. Each figure was drawn to scale on cross-section paper and then the sixteen measurements were made on each figure. These constituted the test scores for the present example. In setting up the dimensions of *Table 6* the numbers were not distributed entirely at random. To do so would tend to make the correlations between the four basic parameters a, b, c, and h approach zero and this would lead to an orthogonal simple structure in which there would be no provocation to investigate a second-order domain. The manner in which the generating conditions of the objects determine the factorial results will be discussed in a later section. *Table 6* was so constructed that, in addition to the four basic parameters, there was also a size factor which functioned as a second-order parameter in determining correlation between the four primary factors in generating the figures.

The product-moment correlations between the sixteen measurements for the thirty-two objects were computed and these are listed in *Table 7*. This correlation matrix was factored by the group centroid method and the resulting factor matrix F is shown in *Table 8*. The fourth-factor residuals are listed in *Table 9*, which indicates that the residuals are vanishingly small. Applying the rotational methods to the configuration, we found the transformation matrix Λ of *Table 10*, which produced the oblique factor matrix V of *Table 11*. In this matrix we are now concerned with all but the last column. When pairs of columns of the factor matrix V are plotted we have the configuration shown in the diagrams of *Figure 6*, in which a simple structure is clearly indicated. The cosine of the angle between the reference vectors is indicated on each diagram of *Figure 6*. These cosines were obtained from the relation $C = \Lambda'\Lambda$ as shown in *Table 12*.

So far in the analysis we have found that four primary factors account for the correlations and this corresponds to the fact that we used four parameters in setting up the trapezoid figures. The four primary factors are correlated as indicated by the obliqueness of the reference axes in the diagrams of *Figure 6*. The next step is to determine the correlations between the primary factors that correspond to the primary reference axes. For this purpose the inverse of the matrix C is computed as shown in *Table 12*. From the diagonal values of this matrix are found the numerical values of the diagonal matrix D, which is also shown in *Table 12*. The inverse of this diagonal matrix is also listed. These numerical values are merely the reciprocals of the entries in D.

In *Table 13* we have the correlation matrix $\boldsymbol{R}_t$ showing the correlations between the primary factors. These are the cosines of the angles between the primary vectors. It can be seen by inspection that this matrix is close to unit rank, which indicates that a single general second-order factor can be postulated to account for the correlations between the primary factors. The saturation of each primary factor with this second-order general factor was determined by one of special formulas for unit rank and the saturations are listed in the column vector r_{tg}. The interpretation is, for example, that the primary factor A has a correlation of .71 with the second-order general factor G. The closeness of the correlation matrix to unit rank is shown by the small side correlations in the residual matrix of *Table 13*. The diagonal values of the residual matrix show that part of the total variance of each primary factor which it does not share with the general second-order factor. If the diagonals of this matrix vanished completely, then the primaries would have their total variance in common and the original reduced correlation matrix for the tests would have been of unit rank.

The saturation of each test with the second-order general factor was determined as a linear combination of the columns of the oblique factor matrix V of *Table 11*. The transformation of equation (18) was used and the numerical values of Ψ_{pg} were listed in *Table 13*. Column G of *Table 11* was then computed by equation (16).

The second-order general factor G can be interpreted in this example as a size factor and it also indicates that in generating the thirty-two figures the four parameters a, b, c, and h were not allowed to take entirely independent values. In other words, the extreme forms of figures either did not occur or else they were used only occasionally. If the four parameters had been allowed to take entirely independent values, then there would have been an appreciable number of figures in which one of these parameters had an unusually small value while some other parameter had some unusually large value. This interpretation of the second-order general factor leads to a consideration of what we shall call *generating parameters*. The present geometrical example illustrates the type of factorial organization that is represented diagrammatically in *Figure 2*. The problem of interpreting the four primary factors can be solved in this case without investigating the second-order domain. But if the correlations between the primary factors show unexpectedly low rank, then this fact can be utilized factorially in gaining further insight into the conditions under which the objects were generated. The four primary factors here identified by the simple structure were the four parameters that were used in setting up the problem.

7. *Generating Parameters*

In addition to the principle of simple structure for the description of each individual object, we may consider an extension of this principle to the problem of describing the manner in which the measured objects were generated. Other things being equal, we should prefer a set of descriptive parameters that give some indication of the conditions that were operative in producing the objects. To the extent that a factor analysis can throw some light on the conditions that were responsible for producing the objects and their measurable characteristics in addition to the description of each individual object, both by some simplifying set of parameters representing causative factors, the factorial methods become even more useful as tools in scientific work.

The numerical values of the trapezoid parameters in *Table 6* defined thirty-two figures of various shapes. The method of constructing the table of four measures for each figure determined whether one or more second-order factors would be present and also whether each of the primaries would be equally or differently represented in the second-order factor. The factorial result could be altered indefinitely by the manner in which the objects were generated in constructing *Table 6*. Since it is the object of factor analysis to reveal the underlying order in the domain, it is an essential part of the numerical example to show that there is a relation between the generating principles and the factorial results.

The first column of the table contains the three linear measurements 1, 2, and 3. Suppose that these were inserted in the column entirely at random. Assume that each column was similarly constructed by distributing a set of measurements entirely at random. Then we should expect zero correlations between the four primaries T_A, T_B, T_C, and T_H. The correlation matrix for the four primary factors would be an identity matrix and it would not be factored because the primary factors would be statistically independent. There would be no second-order factor present.

If, for each one of these thirty-two figures with uncorrelated primaries, we should draw another one similar in proportions but with twice the area and another one with similar proportions but three times the area, then we should have a set of ninety-six figures consisting of three sets that have similar shapes but different sizes. If this new set of ninety-six figures were analyzed factorially with the same battery of sixteen measurements, we should find the same primary factors but they would be correlated. Furthermore, the correlations of the primary factors would all be the same, so that we should have a correlation matrix for the primary factors with uniform side

correlations. The reduced correlation matrix would have unit rank and all of the four primaries would have the same saturation on the second-order general factor. This would be a situation with a second-order general factor which has a uniform effect on all of the primaries. Here again, the factorial result would be determined by the manner in which the objects were generated.

Suppose that a group of persons were asked to draw some trapezoids of arbitrary shapes and that these trapezoids were assembled as a population of figures to be measured and analyzed factorially. Then we should almost certainly introduce a second-order size factor because our subjects would probably unwittingly draw the figures so that the several dimensions of each figure would be at least roughly of the same general order of magnitude. Some of the subjects might draw trapezoids of the general size of, say, five or six inches while other subjects might draw figures only one or two inches across. Very few would produce trapezoids that are one or two inches wide and ten inches tall. In other words, since some subjects would draw big figures and others small ones and since they would probably produce very few extreme figures, there would be strong correlation between the primary factors and these in turn could be analyzed factorially into secondary factors. In this situation the rank of the correlations of the primary factors would probably not be exactly one but the inference could certainly be drawn from the factorial result that secondary factors were operative to produce some big figures and some small ones in addition to the primary parameters that define the individual figures.

The interpretation of the second-order factor as a size factor in the trapezoid example should be distinguished from the size factor that could be chosen as a parameter in the first-order domain. If one of the measurements had been the total area of the trapezoid, it would have been represented by a test vector in the middle of the configuration since it would be affected by all four of the generating parameters that were used and which appeared in the simple structure. The total area test vector could be normalized to a unit vector and it could be used as one of the parameters for describing the trapezoids. It would not be identical with the second-order size factor but they would be closely related. Whether a size factor appears as a first-order factor or as a second-order factor depends on the restrictive conditions under which the figures or objects are produced or selected and also on the selection of measurements for the test battery. It is interesting to note that here the results would indicate either that the thirty-two trapezoids had been systematically selected by some restrictive conditions or else that the objects themselves had been generated un-

der some restrictive conditions.

When the factorial results are clear in both the first-order and second-order domains, inferences can sometimes be drawn concerning the generating conditions that produced the individual parts of the objects. Such inferences can be the basis for formulating hypotheses that can be investigated further either by factorial methods or by more directly controlled experiments.

8. *Incidental Parameters*

So far we have considered the primary factors determined by a simple structure as representing parameters that can be given some scientific interpretation in terms of concepts that are fundamental for the domain in question. In using the simple structure solution which leads sometimes to the second-order domain, we have tried to avoid using arbitrary parameters whose only merit is that they serve in the condensation of numerical data. We have tried to find in the primary factors a set of parameters that not only describe the individual measurements but which also reveal something about the underlying order in the domain. In looking for meaningful parameters of this kind it would be an error to assume that all of the factors have significance that transcends the particular experiment or the particular group of subjects. It would be strange indeed if factor analysis were immune from the distracting circumstances of the particular occasion. The experimenter must try to distinguish that which is invariant and which transcends the particular experimental arrangement or the particular experimental group of subjects from that which is local and incidental to the particular occasion. In factor analysis we are not relieved of this difficult task any more than in other forms of scientific experimentation. In order to focalize attention to this circumstance it might be well to distinguish the primary factors which represent the invariants for which we are really looking from those primary factors which, though genuine as regards the explanation of the test variances, are local and of significance only for the experimental group or the particular occasion. *Primary factors which characterize only a particular experimental group or a particular situation may be called incidental factors to distinguish them from the invariants which are normally the object of scientific experimentation.* Incidental factors may appear in the first-order or in the second-order domain.

A few examples will serve to illustrate the manner in which incidental factors may appear as primaries in factorial analysis. In addition to the primary factors that would be found in different groups of subjects, we might find primary factors that are unique for the par-

ticular occasion. Suppose that an exceptionally good examiner who is skilled in obtaining good rapport with the subjects should give a part of the test battery to a part of the experimental group. A primary factor might appear for this group of tests and the investigator might be at a loss to explain it because he would be thinking about the nature of the tests and he would try to find something common in the psychological nature of these tests. It might not occur to him that this is the very group of tests that were administered by the experienced examiner. Such a factor would probably be left without interpretation in the final results or the interpretation might be one that would not be sustained in a subsequent experiment with different subjects and different examiners. Incidental factors are almost certainly present in every study. Hence the investigator should feel free to leave without interpretation those primary factors which do not lend themselves to rather clear scientific interpretation. Even then the interpretation should be at first in the nature of a hypothesis to be sustained if possible by subsequent factorial studies. The fact that all of the variances are not adequately accounted for in the interpretation has led some students to conclude that the whole result should be discarded, but such is not the case. It is quite possible to make an important discovery concerning the primary factors that are operative in an experiment even though the major part of the common factor variances remains unexplained. It is assumed, of course, that such a finding could be sustained by the construction of new tests with prediction as to how they should behave factorially in new groups of differently selected subjects.

In one factorial study it was found that a primary factor was common to a set of tests that were given by the projector method with individual timing for each response. The interpretation of such a factor was uncertain. Some psychological function might be involved in the projector tests which was absent from the other tests, but the explanation might also be that some motivational condition was common to the projector tests that was absent from the other tests and which would be of only incidental significance as far as the major purposes were concerned.

Suppose that one of the examiners misunderstands the time limits for a set of tests and that he gives the shorter time limits to a part of the group of subjects for some of the tests. A factor might appear under certain circumstances that would be incidental and of no fundamental significance, but the primary factors that are significant might still be revealed. An unexpected interruption in a school examination such as fire drill, a street parade, or the expectancy of an important school event may act to introduce incidental factors.

One of the most important sources of incidental factors is to be found in the selective conditions. If a group of subjects is selected because of qualification in a composite of two or more tests, the unique variances of such selective tests combine to form one or more incidental common factors which would have remained a part of the unique variance if the selective conditions had not been imposed. The correlations between the factors are determined in large part by the selective conditions. If a group of subjects is selected because of certain test qualifications, it is to be expected that the primary factors will show correlations between factors that are different from the correlations between the same factors in an unselected population. It must not be assumed that the factors are different just because they correlate differently in different populations. This effect is well known with physical measurements, height and weight with intelligence, for example, whose intercorrelations are determined in large part by the selective conditions. These changes do not affect the identity of the factors. An incidental factor which is introduced by conditions of selection may be trivial or it may be of significance, depending on the nature of the unique variances which are introduced into the common factors by the selective conditions.

It should be remarked that in a well planned factorial experiment the incidental factors are usually of secondary importance in comparison with the variance that is assignable to the principal primary factors for which an experiment was planned. When one or more primary factors have relatively small variance and do not seem to lend themselves to clear interpretation, they should be reported without interpretation. Some reader of such a report may find a fruitful hypothesis for it, or the factor may be of only incidental significance.

These few examples will serve to call attention to the fact that not all the primary factors can be expected to have meaning in the fundamental sense of representing functional unities whose identity transcends the particular group of subjects and the experimental conditions of any particular occasion. It does not follow that incidental factors are in any sense artifacts. They may represent genuine factors that were operating to produce the observed individual differences but their significance may not extend beyond the particular occasion. In that sense they are irrelevant to the purposes of the experimenter even though they are valid as factors which can sometimes be identified.

An interesting application of second-order factors is an attempt to reconcile three theories of intelligence, namely, Spearman's theory of a general intellective factor, Godfrey Thomson's sampling theory

with what he calls "sub-pools," and our own theory of correlated multiple factors which are interpreted as distinguishable cognitive functions. The tetrad differences vanish when there are no primary factors common to the four tests of each tetrad, the correlations being determined only by the general second-order factors. This application of second-order factor theory will be the subject of a subsequent paper.

TABLE 1
Orthogonal Factors

	A	*B*	*C*	*D*	*E*
1	x				x
2	x	x			x
3			x		x
4	x	x			x
5				x	x
6			x	x	x
7			x		x
8				x	x

TABLE 2
Correlated Factors

	A	*B*	*C*	*D*	*E*
1	x				
2	x	x	x		
3	x				
4		x			
5			x		x
6		x	x	x	
7				x	
8				x	x
9					x

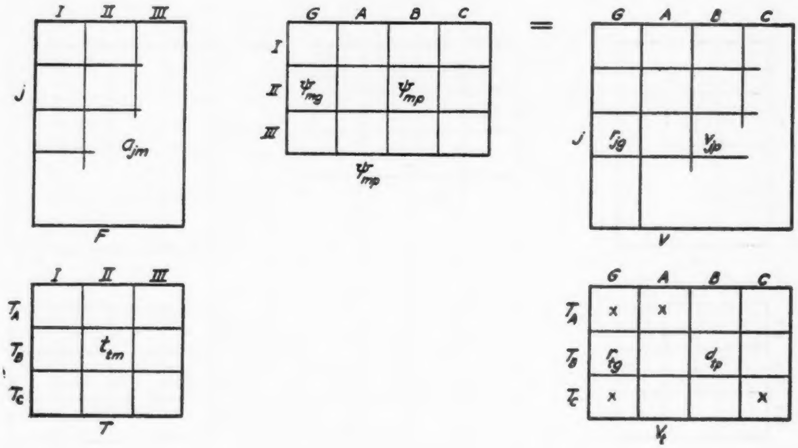
	G
A	x
B	x
C	x
D	x
E	x

TABLE 3
Second-Order Domain
of Figure 3

	P	*Q*
A		
B	x	x
C	x	
D	x	x
E	x	
F	x	
G	x	

TABLE 4

Case 1. Transformation from $\boldsymbol{F}$ to $\boldsymbol{V}$ including the column vector G



Case 2. Transformation from $\boldsymbol{F}$ to $\boldsymbol{U}$ including the group factors and the column vector G

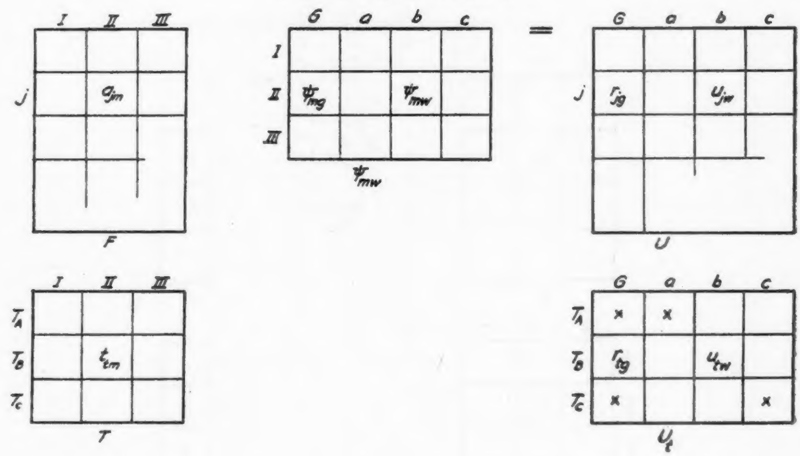
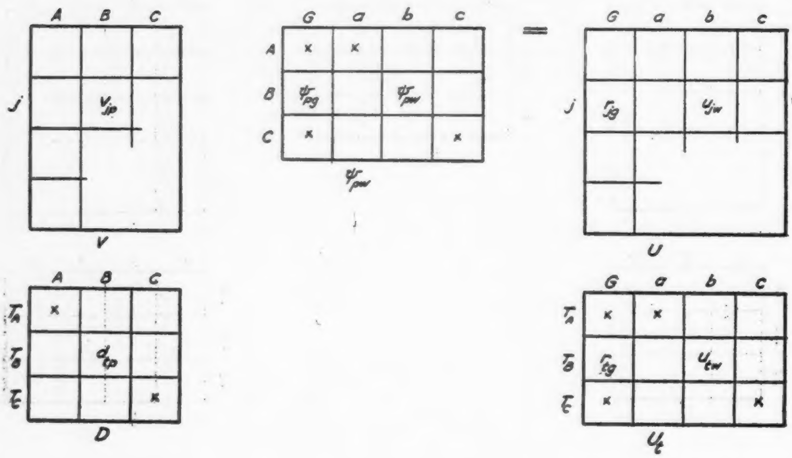


TABLE 5

Case 3. Transformation from *V* to *U* including the group factor and the column vector *G*



Case 4. Transformation from *V* to column vector *G*

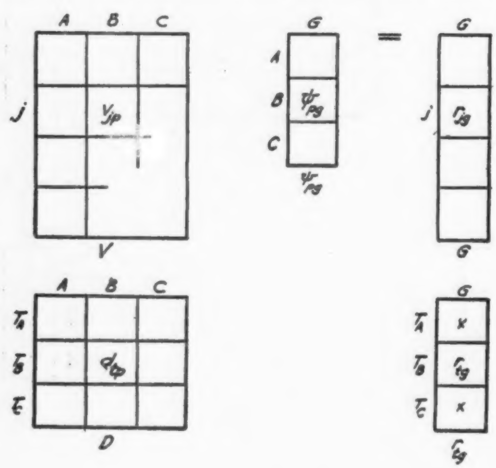


TABLE 6
Trapezoid Parameters

	a	b	c	h
1	1	2	1	2
2	1	2	1	4
3	1	2	3	2
4	1	2	3	4
5	1	3	1	2
6	1	3	1	4
7	1	3	3	2
8	1	3	3	4
9	2	2	1	2
10	2	2	1	4
11	2	2	3	2
12	2	2	3	4
13	2	3	1	2
14	2	3	1	4
15	2	3	3	2
16	2	3	3	4
17	2	3	3	3
18	2	3	3	5
19	2	3	5	3
20	2	3	5	5
21	2	4	3	3
22	2	4	3	5
23	2	4	5	3
24	2	4	5	5
25	3	3	3	3
26	3	3	3	5
27	3	3	5	3
28	3	3	5	5
29	3	4	3	3
30	3	4	3	5
31	3	4	5	3
32	3	4	5	5

TABLE 7

Correlation Matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1.00	.50	.50	.32	.29	.58	.72	.49	.58	.45	.31	.66	.53	.76	–.35	.11
2	.50	1.00	.50	.32	.36	.42	.57	.49	.74	.67	.33	.54	.64	–.16	–.14	–.23
3	.50	.50	1.00	.32	.52	.42	.88	.82	.90	.45	.30	.78	.75	.19	–.84	–.72
4	.32	.32	.32	1.00	.95	.96	.65	.80	.61	.91	.98	.78	.82	.12	–.22	–.15
5	.29	.36	.52	.95	1.00	.90	.75	.90	.75	.90	.94	.84	.89	.05	–.37	–.31
6	.58	.42	.42	.96	.90	1.00	.78	.83	.70	.92	.94	.86	.86	.34	–.29	–.09
7	.72	.57	.88	.65	.75	.78	1.00	.95	.95	.74	.64	.95	.91	.39	–.69	–.46
8	.49	.49	.82	.80	.90	.83	.95	1.00	.93	.83	.78	.94	.95	.19	–.64	–.52
9	.58	.74	.90	.61	.75	.70	.95	.93	1.00	.79	.60	.90	.93	.11	–.64	–.57
10	.45	.67	.45	.91	.90	.92	.74	.83	.79	1.00	.90	.83	.90	.01	–.22	–.21
11	.31	.33	.30	.98	.94	.94	.64	.78	.60	.90	1.00	.77	.80	.11	–.12	–.09
12	.66	.54	.78	.78	.84	.86	.95	.94	.90	.83	.77	1.00	.97	.34	–.59	–.39
13	.53	.64	.75	.82	.89	.86	.91	.95	.93	.90	.80	.97	1.00	.12	–.52	–.44
14	.76	–.16	.19	.12	.05	.34	.39	.19	.11	.01	.11	.34	.12	1.00	–.28	–.34
15	–.35	–.14	–.84	–.22	–.37	–.29	–.69	–.64	–.64	–.22	–.12	–.59	–.52	–.28	1.00	–.76
16	.11	–.23	–.72	–.15	–.31	–.09	–.46	–.52	–.57	–.21	–.09	–.39	–.44	.34	.76	1.00

TABLE 8

Orthogonal Factor Matrix F

	I	II	III	IV
1	.57	.44	.63	.16
2	.59	—.03	—.01	.59
3	.79	—.46	.38	.03
4	.81	.35	—.42	—.25
5	.88	.13	—.38	—.24
6	.87	.46	—.14	—.12
7	.96	—.02	.30	.00
8	.98	—.07	—.02	—.10
9	.95	—.19	.10	.27
10	.88	.25	—.36	.17
11	.78	.39	—.44	—.14
12	.97	.09	.10	—.05
13	.98	.01	—.09	.06
14	.21	.47	.65	—.38
15	—.60	.50	—.44	.37
16	—.46	.77	.02	.08

TABLE 9

Distribution of Residuals

Dev.	f
.00	50
.01	94
.02	46
.03	30
.04	8
.05	2
.06	2
.07	0
.08	4
.09	2
.10	2

$N = 240.$

TABLE 10
Transformation Matrix Λ

	A	B	C	H
I	.07	.12	.39	.53
II	.70	—.01	—.81	.35
III	.71	—.32	.28	—.64
IV	—.01	.94	—.34	—.44

TABLE 11
Oblique Factor Matrix V

	A	B	C	H	G
1	.79	.01	—.01	—.02	.68
2	.01	.63	.05	.05	.63
3	.00	.01	.78	.00	.73
4	.01	—.01	.00	.93	.46
5	—.11	.00	.21	.86	.52
6	.28	.03	—.03	.76	.62
7	.27	.02	.47	.31	.84
8	.01	.03	.47	.55	.74
9	.00	.34	.46	.25	.85
10	—.02	.38	—.02	.71	.64
11	.02	.10	—.09	.91	.47
12	.20	.04	.35	.50	.78
13	.01	.20	.33	.55	.76
14	.81	—.54	.01	.03	.27
15	—.01	.41	—.89	—.02	—.49
16	.52	.01	—.82	—.02	—.31

TABLE 12

Matrix $C = \Lambda'\Lambda$

	A	B	C	H
A	1.00	—.24	—.34	—.17
B	—.24	1.00	—.35	—.15
C	—.34	—.35	1.00	—.11
H	—.17	—.15	—.11	1.00

Matrix D_{tp}

	A	B	C	H
T_A	.808			
T_B		.808		
T_C			.786	
T_H				.912

Matrix C^{-1}

	A	B	C	H
A	1.53	.73	.83	.46
B	.73	1.53	.84	.44
C	.83	.84	1.62	.45
H	.46	.44	.45	1.19

Matrix $D^{-1}{}_{tp}$

	T_A	T_B	T_C	T_H
A	1.237			
B		1.237		
C			1.273	
H				1.091

TABLE 13

Correlation Matrix $R_t = D_{tp} C^{-1}_{pq} D_{pt}$

	T_A	T_B	T_C	T_H
T_A	1.00	.48	.53	.34
T_B	.48	1.00	.53	.33
T_C	.53	.53	1.00	.32
T_H	.34	.33	.32	1.00

Column Vector r_{tg}

	r_{tg}
T_A	.71
T_B	.70
T_C	.73
T_H	.45

Residuals $= R_t - r_{tg} r'_{tg}$

	T_A	T_B	T_C	T_H
T_A	.50	—.02	.01	.02
T_B	—.02	.51	.02	.01
T_C	.01	.02	.47	—.01
T_H	.02	.01	—.01	.80

Column Vector $\Psi_{pg} = D^{-1}_{tp} r_{tg}$

	G
A	.878
B	.866
C	.929
H	.491

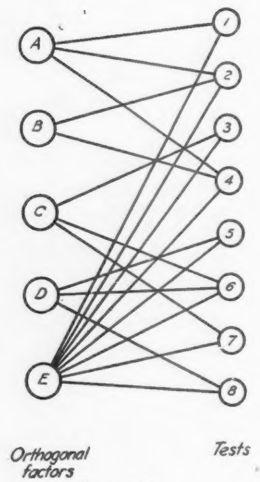


FIGURE 1

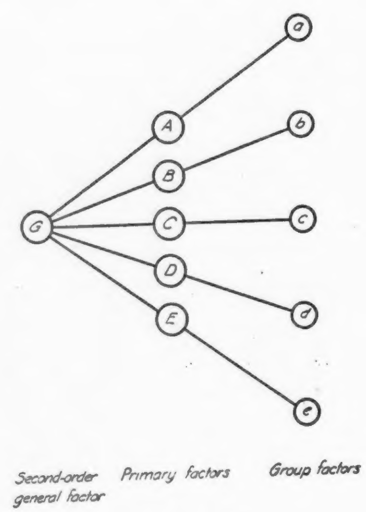


FIGURE 2

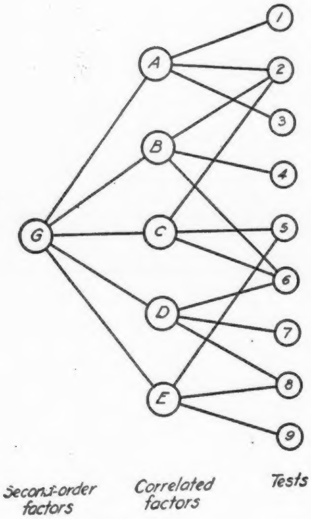


FIGURE 3

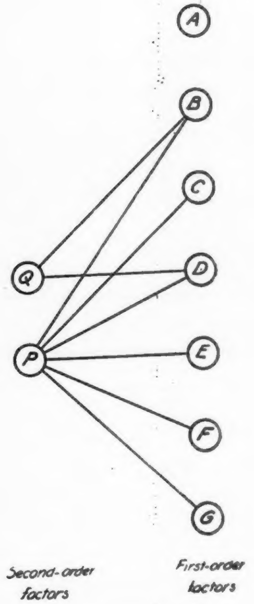


FIGURE 4

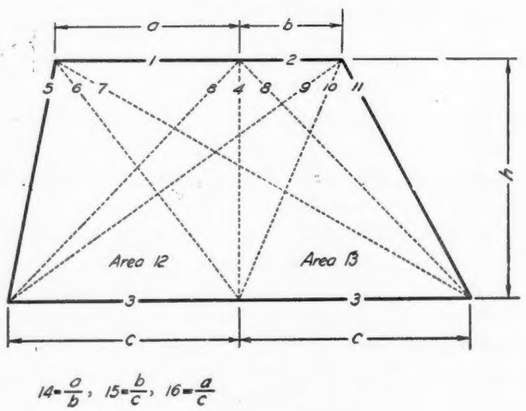


FIGURE 5

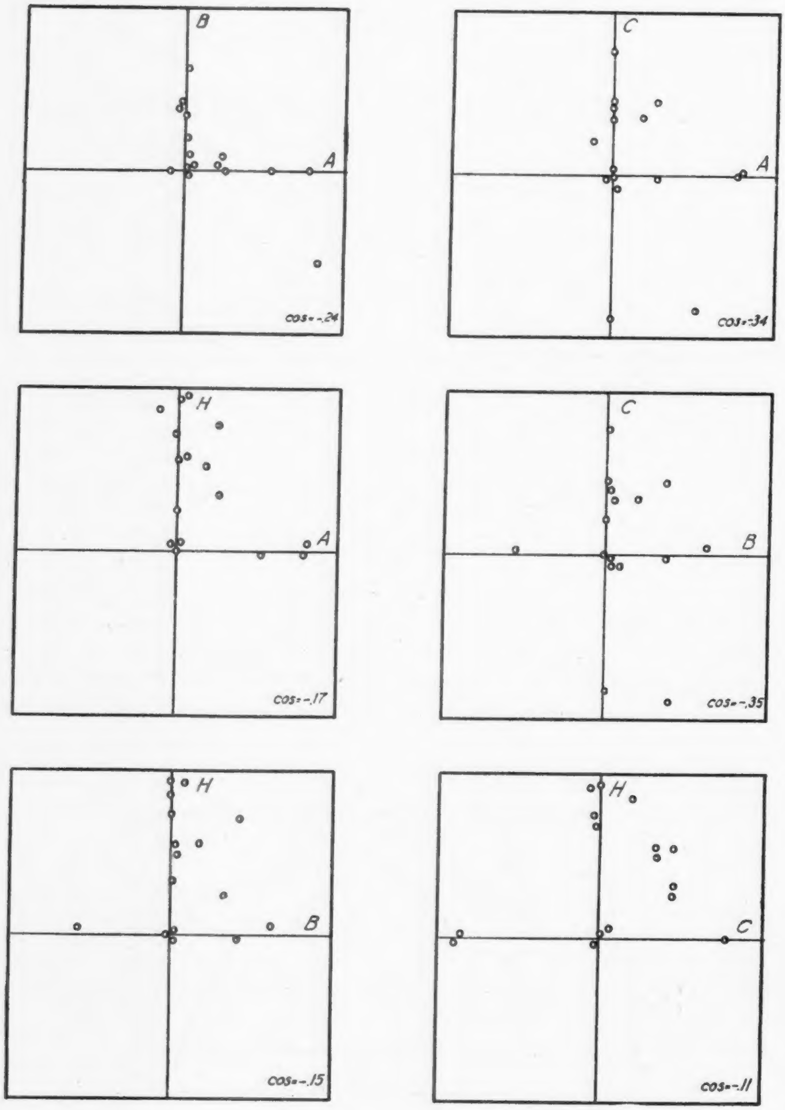


FIGURE 6

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TECHNIQUE FOR WEIGHTING OF CHOICES AND ITEMS ON I.B.M. SCORING MACHINES

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A technique has been developed which permits the weighting of responses of test items on the I. B. M. scoring machine on the initial scoring, heretofore impossible. This is done by making the length of the response lines on the answer sheet longer or shorter as weights are needed. It is anticipated that this method will prove useful wherever differential weighting serves to increase the validity of tests.

Introduction

In the past, weighting of answers to items on a questionnaire or test by scoring machines has been very difficult if not prohibitive. In a single run through the I.B.M. scoring machine, weights of -1, 0, and $+1$ only can be used. The addition of merely one more weight requires an extra run through the machine. In two runs a range of nine weights can be obtained, while a range of nineteen weights can be obtained in three runs. The technique here permits the weighting of responses to test items without requiring additional runs through the machine at least for the same marks on the answer sheet. In order to understand this procedure, a brief explanation of the I.B.M. scoring machine is needed.

I. B. M. Procedures

To obtain any score on the machine, a separate answer sheet is provided on which the testee fills in his choices with a graphite lead pencil. Each marked-in space on the answer sheet completes a circuit across one set of contact plates on the sensing unit. This current goes through a set of resistors to a key-set-unit. There is a pin on the key-set unit corresponding to each set of contact plates on the sensing unit.

When a blank stencil prepared for scoring is inserted in the front leaf of the stencil holder and pressed against the key-set-unit by means of an adjustment of the frame, the pins are pressed all the way in. When a graphite mark occurs on the answer sheet to complete the circuit, the dial will register a minus weight. When a stencil

is punched, it allows the pins to go all the way through. If a mark occurs on the answer sheet in this position, the weights will be plus. If the pin is pushed only halfway in, the item is entirely eliminated. This is done by putting a stencil in the front leaf of the holder with the desired eliminated items and the plus items punched. In the back leaf of the holder, a stencil is inserted with only plus items punched. This allows the pins of the eliminated items to go only halfway, while the pins of the positively weighted items go completely through.

Weighting

In weighting responses to tests of specific knowledge (physics, mathematics, or any factual material), the validity of each item must be established before proper weights can be assigned. It is necessary that each answer space be printed uniformly and that the subject's response line, regardless of the weight assigned by the test constructionist, fill the complete space. This is imperative in order that the testee be unaware that his responses are weighted. The weighting is effected by the intricate manipulation of the key-set-unit.

This technique of weighting can be advantageously employed in perceptual and judgment fields, where degrees of correctness are of importance. Here, too, it is necessary that answer spaces be uniform and that each response cover the complete space.

On questionnaires and on tests of personality or emotion, in which the subject's intensity of response is desired, the procedure is reversed and the subject assigns his own weights by the length of his pencil mark. Regardless of the range of weights used, each question should have response spaces of uniform length.

Weighting the Individual Items of a Test with Specific Weights for Each Item

Presuppose a test where the validity has been established for each response and a range of five weights has been decided on. An answer sheet such as that shown in Figure 1, A, will be used. On this answer sheet, the response line is twice the length of the presently used form. Thus, it will be seen that the testee must mark a line for each item that is long enough to extend over the space occupied by two sets of contact plates. Let us assume that the assigned weights for question one are plus two for response "A," plus one for "B," zero for "C," minus one for "D," and minus two for "E." Following the principles explained in preceding paragraphs, it is necessary to punch out on the stencil to go into the front leaf all those choices which are weighted plus and those eliminated as shown in Figure 2, A. On the example used, A, B, and C will be punched as shown. Since choice D is weight-

ed only minus one (half the possible minus weight), it is necessary to eliminate the other potential half. On the stencil that is to be placed in the back leaf of the stencil holder, the final plus weights are determined (as shown in Figure 2, B). Since A is weighted plus two, these two holes are punched. Since B is weighted plus one, only one hole is punched for it. When the stencil holder is pressed against the pins, the key-set-unit will be modified as shown in Figure 3.

Answer Sheets for Multiple-Choice Tests

Most aptitude, achievement, and perceptual tests that are electrically scored are of the multiple-choice type. Therefore, each response space must be uniform.

It is possible to employ currently used I.B.M. answer sheets, but this would require mimeographing over the answer sheets so that the length of the space gives the appropriate maximum plus or minus weights. Because of the exactness required for each answer sheet, however, this method is not recommended. It is far more desirable to have special forms printed for each range of weights. It would also be desirable to print the spaces on both sides so that more questions may be answered on one answer sheet.

If a range of five weights (-2, -1, 0, $+1$, $+2$) is desired, an answer sheet with spaces printed to cover two sets of contact plates must be printed (Figure 1, A). With a five-choice item, a possible total of seventy questions may be answered on one side. When a range of seven weights (-3, -2, -1, 0, $+1$, $+2$, $+3$) is needed, the answer sheet must be printed so that each space covers three sets of contact plates (as shown in Figure 1, B). If this test has five-choice items, a total of fifty questions on one side of the answer sheet is possible.

For all practical purposes, to extend the weights beyond seven, it is necessary to reduce the number of choices on any one question. It is hardly possible to expect the subject to mark a three-inch line both clearly and darkly enough to be picked up entirely by the machine.

Since a given horizontal row contains only fifteen sets of contact plates and a vertical row only five sets, an item with a weight range of nine or eleven can have only three choices. A question with a range of twelve weights or above is limited to two choices. Regardless of the particular weight used, the printed answer spaces must be constant, and the subject must fill each in completely. Thus, in the case of some weights, the full fifteen horizontal spaces cannot be utilized.

On tests where intensity of response is the important factor and the subject assigns his own weight, uniformity of the answer space

for each item is desirable. However, the subject must be impressed with the fact that he is doing his own weighting by means of the number of answer spaces he covers with his graphite pencil.

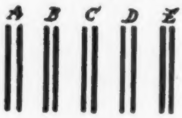
Conclusion

Weighted scoring of tests by hand, as well as the old method of machine weighting, is not practical, especially when tests are given to large groups at frequent intervals. It is anticipated that the method described in this report will prove very useful wherever differential weighting serves to increase the validity of tests.

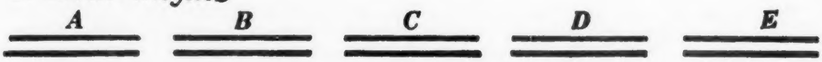
FIGURE 1: Answer Sheets

In the new form the spaces for item one are the same as those covered by items one and two on the standard answer sheet. (No. 838).

A. Five Weights



B. Seven Weights



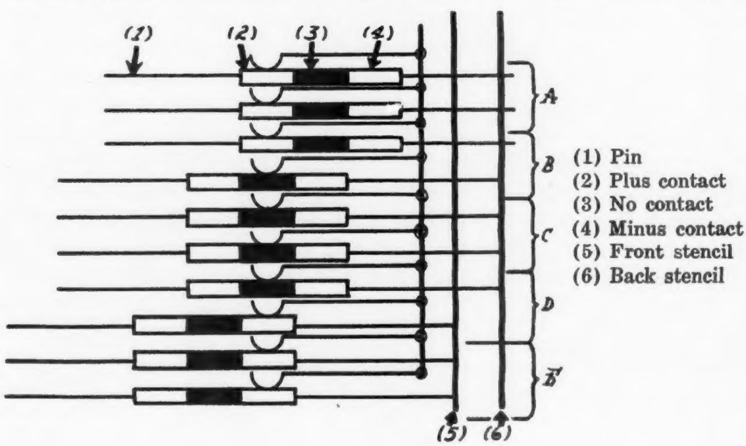
Answer sheet similar to the fifteen-choice answer sheet (No. 881).
Choice A covers same space as did choices A, B, C, on form No. 881 .
Choice B covers D, E, and F on old forms, etc.

FIGURE 2: Stencil—First and Second Items

A. Front Stencil B. Back Stencil

FIGURE 3: Position of Pins When Pushed In

The letters in front of the pins represent the pins necessary for each choice and what position each must be in to give the desired weights in the example.



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FACTORIAL DESIGN IN THE DETERMINATION OF DIFFERENTIAL LIMEN VALUES*

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This paper discusses the application of the principles of factorial design to an experiment in psychology. For the purpose of illustrating the principles, a simple experiment was designed dealing with the determination of the differential limen values of subjects for weights increasing at constant rates. The factorial design was of the type: 4 rates x 7 weights x 2 sexes x 2 sights x 2 dates. The appropriate statistical analysis for this type of design is the analysis of variance. The mathematical formulation of the problem was specified and the appropriate solution for the specific problem was obtained. Greater precision results from this type of design, in comparison with the traditional psychological experiment dealing with a single factor, in that measures are obtained of the effect of each of a number of factors together with their interactions.

Mathematicians have, at least since the time of Euler, been interested in combinatorial problems dealing with the arrangement of a finite number of things in sets, or patterns, or configurations, satisfying specified conditions. However, it remained for Fisher and his associates to show how combinatorial principles can be put to the greatest practical use in the designing of experiments. Beginning in 1922, Fisher laid down principles of designing experiments which revolutionized the techniques of agricultural trials and have changed the course of the design of agricultural experiments throughout the world. In his book, *The Design of Experiments*,† now in its third edition, 1942, Fisher has laid the framework of scientific inference and has developed the principles of experimentation which are finding increasing application in many fields of science. Replication, randomization, and local control, as fundamental requirements of a self-contained experiment, are found to be principles of general utility wherever the basic materials are variable. The difficulties met with in one field are not identical, but many are similar, to those in other fields. The solutions arrived at in one field are often of material assistance in another. In so far as fields differ fundamentally, new techniques are required and these can be developed only in direct contact with the obstacles themselves.

* This is one of a number of studies on modern principles of experimental design. For the research grant to finance these studies, grateful acknowledgment is given to the Graduate School, the University of Minnesota.

† Fisher, R. A. The design of experiments. London: Oliver and Boyd, 1934.

The new types of design are based upon ideas which differ sharply from earlier ones as to the number of enquiries to be included in a single experiment. The modern experimenter usually combines several single lines of enquiry in the same experiment. The traditional experimenter believes that an experiment must be simple and stresses the importance "of varying the essential conditions *only one at a time.*" Among the experimental designs that have been so far developed, one in which several different problems are included in the same experiment is the factorial design. In this design, all the factors to be examined are varied concurrently in all possible combinations. The chief advantages of this type of design over the traditional experiment designed to examine a single question, or a single factor, reside in its greater efficiency and comprehensiveness. These advantages are attained through the fact that in a factorial experiment every trial contributes to the answering of every question with the same precision as though the whole experiment had been given over to any one of them and that, in addition to measuring the effects of each of the single factors the measures of the effects of the interactions of all combinations of factors are obtained with the same precision. The latter advantage is especially great since with separate single-factor experiments no measure at all is attainable of the interaction of the different factors. A third distinct advantage of factorial design lies in the fact that this plan gives results of wider applicability than do single experiments, since the exact standardization of experimental conditions prescribed for the traditional experimental design provides information only in respect to a narrowly restricted set of conditions. In the factorial design, the ingredients may be varied, i.e., applied at different levels, while in the single factor experiments standardization requires that the other factors be kept constant. Rarely does this standardization of conditions go beyond that attained by a more or less arbitrary definition.

The factorial design promises to be of especial value in the design of experiments in psychology where standardization of conditions is difficult, if not often impossible, and where the measure of interacting factors may be of even greater significance than of the direct effect. It was to gain at first hand an insight into the nature of the modern principles of design and to explore their possibilities that the investigation reported here was carried out. It is futile to dogmatize about the possibilities of the newer ideas. What they can do can only be discovered by trial. The report is divided into two parts. In Part I we attempt to exhibit the principles of factorial design in use by presenting the design and the analysis of the results of an experiment in psychology. The existing literature on factorial designs is

concerned rather with planning more or less complex experiments and with the arithmetical procedures in treating the results than with the mathematical theory underlying the method.* Part II, which may be omitted by the non-mathematical reader, is devoted to the development of the mathematical foundation fundamental to the appropriate statisticial analyses. Command of the procedure to be followed in attacking problems of this kind should enable the statistician to develop and analyze designs for attacking new problems.

PART I

The psychological experiment to which the method of factorial design was applied consisted in determining the differential limen of subjects for weights increasing at constant rates. An apparatus, was set up by which it was possible to increase continuously certain standard weights at a specified rate. An aluminum pail was attached by a hook to a lever mounted on a tripod and the system placed in equilibrium by an adjustable screw controlling a movable weight. At one end of the lever arm there was a circular ring into which the subject placed his index finger, by which he lifted the pail containing a weight to which additions were being made at a constant rate of increase. A graduate of 1000 cc capacity was mounted in a frame and filled with water. Four glass tubes of different bores calibrated to four different rates were fitted into the bottom of the graduate. These glass tubes were connected with rubber tubes entering into the bottom of the aluminum pail through a valve that controlled the flow of water to prevent the impact and sound of flowing water from distracting the subject. Stop-cocks regulated the flow of water into the pail. The four rates of flow were established as 50, 100, 150, and 200 cc per 30 seconds. Seven different standard weights were used — 100, 150, 200, 250, 300, 350, and 400 grams. Each standard weight was combined with each of the standard increasing rates so that there were in all 28 weight-rate combinations.

The experimental subjects were four men and four women. Two of each sex were normally sighted; the other two were congenitally blind. The two normally sighted males were 21 and 25 years of age. The two congenitally blind males were of the same respective ages. The two normally sighted females were each 33 years old, the same age as each of the congenitally blind females.

* Two very informative publications on applications of factorial design, especially the second in relation to the problem discussed here, are:
1. Yates, F. Imperial Bureau of Soil Service. Technical Communication No. 35. Imperial Bureau of Soil Science: Harpenden, England, 1937. 95 pp.
2. Mahalanobis, P. C., and Nair, K. R. *Sankhya, The Indian Journal of Statistics*, 1941, 5, 285-94 [Statistical Society, Calcutta].

Two preliminary trials were run previous to the main experiment to get the subjects accustomed to the apparatus and experimental technique. The observer O is standing and if normally sighted is blindfolded. The lever arm is placed in the zero position with the aluminum pail in balance. O's index finger is placed in the circular slot at the end of the lever arm (either his right or left index finger, dependent upon his handedness.) The lever is released and O is asked to raise and lower the lever arm from two to three inches and to sense the heaviness. A 50-gram weight is then placed in the pail, and O is asked to sense the heaviness, indicating by saying "ready." The rubber tube with rate 200 cc per 30 sec. is opened and the water allowed to flow into the pail. O indicates by saying "now" when he senses the just noticeable heavierness in the lifted lever arm. At this instant the stop-cock shuts off the flow of water. The experimenter, E, pours off the water from the pail into a graduate and measures to the nearest 0.1 cc. The number of cc of water is recorded as the difference limen value in grams. The magnitude of stimulus that corresponds to a least sense interval from some positive point on the sense scale is defined as the difference limen. The 50-gram weight is then removed and combined weights of 450 grams are placed in the pail. The graduate is filled to the 1000 cc mark. The rubber tube with the 50 cc rate per 30 seconds is then opened and water allowed to flow into the pail. O again indicates the moment of just noticeable heavierness.

The main experiment was then carried out. Five differential limen values (D.L.) were determined for each 0 on each of the 28 rate-weight combinations. The order of presentation of each combination was established in advance by the use of Fisher and Yates' set of random sampling numbers. Catch stimuli were also randomly introduced to check the reality of the O's response. The entire experiment was repeated on each subject after an interval of one week. There were, therefore, 280 D.L. values for each 0. The arithmetical mean of the 5 observational values for each individual-rate-weight-date combination was used in the analysis of variances. The data conformed to a 4 x 7 x 2 x 2 x 2 factorial design, that is, the combination of 4 rates, 7 weights, two sights, two sexes, and two dates. The analysis of the experimental observations follows.

The Total System of Interactions

We proceeded first to test the null hypotheses dealing with the interactions. The policy is rather often followed, on the basis of findings in agricultural experiments, to assume that higher-order interactions are not significant. It is recommended here that interactions

of all orders should be tested when the numerical measure of the interaction can be obtained from the data. In our problem there were ten measures of interactions of the first order, ten measures of interactions of the second order, five measures of triple interactions, and one of quadruple interaction. The tests of significance of the twenty-six interactions are recorded in Table 1. From these tests it was found that the following interactions were significant:

sex x sight x rate
sex x sight
sex x rate
sight x rate
sight x weight (doubtful).

The significant as well as the doubtful interactions were retained as specific components in the analysis of variance table. The interactions shown to be statistically insignificant were included in experimental error. We thus found that the 447 independent comparisons among the 448 different differential limen values could be resolved into the 11 components as shown in Table 2. The tests of signifiance resulted in the following conclusions:

significant main effects:	sex, sight, weight, and rate
significant second-order interactions:	sight x sight x rate
significant first-order interactions:	sex x sight
	sex x rate
	sight x weight
	sight x rate

It is important to note that there was no significant difference between dates and that no interaction including date was significant. This demonstrates that the observations were consistent among themselves.

The Main Effects of Weights

The main effect of weights was found to be significant ($F = 3.18$, Table 2). This indicates that the mean D.L. value varied significantly with the weight. The relation between the D. L. values and weights can be expressed by the linear regression equation

$$\hat{Y} = 24.8554 - .983921x\,,$$

where $\hat{Y}$ is the predicted D.L. value for a given value of the weight, $x = \dfrac{W - 250}{50}$. For this determination it is convenient to use orthogo-

TABLE 1

Tests of Significance of Interactions

Source of Variation	D.F.	Sum of Squares*	Mean Square	F†	Test of Hypothesis‡
Error	224	34,438	154	—	—
Sex x Sight x Weight x Rate x Date	18	270	15	—	Accepted
Sex x Sight x Weight x Rate	18	320	18	—	Accepted
Sex x Sight x Weight x Date	6	379	63	—	Accepted
Sex x Sight x Rate x Date	3	60	20	—	Accepted
Sex x Weight x Rate x Date	18	538	30	—	Accepted
Sight x Weight x Rate x Date	18	205	11	—	Accepted
Sex x Sight x Weight	6	1,406	234	1.52	Accepted
Sex x Sight x Rate	3	2,216	739	4.80	Rejected
Sex x Sight x Date	1	270	270	1.75	Accepted
Sex x Weight x Rate	18	215	12	—	Accepted
Sex x Weight x Date	6	637	106	—	Accepted
Sex x Rate x Date	3	61	20	—	Accepted
Sight x Weight x Rate	18	654	36	—	Accepted
Sight x Weight x Date	6	340	57	—	Accepted
Sight x Rate x Date	3	14	5	—	Accepted
Weight x Rate x Date	18	527	29	—	Accepted
Sex x Sight	1	14,130	14,130	91.75	Rejected
Sex x Weight	6	405	68	—	Accepted
Sex x Rate	3	5,720	1,907	12.38	Rejected
Sex x Date	1	9	9	—	Accepted
Sight x Weight	6	2,089	348	2.26	Remain in Doubt
Sight x Rate	3	2,083	694	4.51	Rejected
Sight x Date	1	4	4	—	Accepted
Weight x Rate	18	391	22	—	Accepted
Weight x Date	6	488	81	—	Accepted
Rate x Date	3	61	20	—	Accepted

* For the calculation of sum of squares, see Table 3 in Part II.

† Where $F = \dfrac{\text{Mean square of a tested variation}}{\text{Mean square of error or residual}}$

By referring to Snedecor's tables of F (See Snedecor, G. W., Statistical methods applied to experiments in agriculture and biology. Iowa: Collegiate Press, 1938, pp. 174-77), we may use the following four rules in testing the hypothesis: (a) reject the hypothesis tested, if the calculated values of F is greater than the 1% point given in the tables; (b) accept the hypothesis tested, if the calculated value of F is less than the 5% point given in the tables; (c) remain in doubt, if the calculated value of F lies between the 5% and 1% points given in the tables; (d) in the event that the mean square of error or residual is greater than the mean square of a tested variation, then we simply accept the hypothesis without calculating the value of F.

These conventions are always true throughout the following tables.

‡ The hypothesis tested is a null hypothesis concerning the variation in the same row. For instance, the hypothesis regarding sex x sight x weight x rate x date is that there is no significant interaction between sex, sight, weight, rate, and date. To give another example, the hypothesis regarding sex x sight is that there is no significant interaction between sex and sight.

TABLE 2
Complete Analysis of Variance of "D.L" Values

Source of Variation	D.F.	Sum of Squares*	Mean Square	F	Test of Hypothesis†
Residual	419	41,692	100	—	—
Sex x Sight x Rate	3	2,216	739	7.39	Rejected
Sex x Sight	1	14,130	14,130	141.30	Rejected
Sex x Rate	3	5,720	1,907	19.07	Rejected
Sight x Weight	6	2,089	348	3.48	Rejected
Sight x Rate	3	2,083	694	6.94	Rejected
Sex	1	31,534	31,534	315.34	Rejected
Sight	1	11,810	11,810	118.10	Rejected
Weight	6	1,909	318	3.18	Rejected
Rate	3	51,072	17,024	170.24	Rejected
Date	1	116	116	1.16	Accepted
Total	447	164,371			

* The sum of squares for residual is simply the sum of the sums of squares for error and all insignificant interactions which are indicated in Table 1. See Table 3 in Part II for the calculation of sums of squares for all the other variations.

† The hypothesis tested is a null hypothesis regarding the variation in the same row. For instance, the hypothesis regarding sex is that there is no significant difference between sex-means.

nal polynomials.* The linear coefficient was significant; both the quadratic and cubic coefficients were insignificant. The analyses of variance for testing goodness of fit is given in Table 3.

The reader who is not familiar with the meaning of linear, quadratic, and cubic terms, is recommended to refer to Goulden's *Methods of Statistical Analysis*, 1939, pp. 166-169. To state it here very simply, if we have only one degree of freedom regarding the tested variation, such as sex and date in our problem, there is only linear relationship between the two levels of variation and the "D.L." values. On the other hand, if we have more than 2 degrees of freedom or more than 3 levels of variation, then we can separate them into component parts: linear, quadratic, cubic, and so on, that are mutually independent. Usually we are well satisfied to calculate up to the cubic term even if we have more than 3 degrees of freedom or more than 4 levels

* To obtain the equations in Tables 3, 4, 5, 8, 11, and 14 the readers are recommended to refer to:

Goulden, C. H. Methods of statistical analysis. New York: John Willey, 1939, p. 219-246.

Anderson, R. L. and Houseman, E. E. Tables of orthogonal polynomial values extended to $N = 104$. Agricultural Experiment Station, Iowa State College of Agriculture and Mechanic Arts, Research Bulletin 297, 1942, pp. 595-606.

TABLE 3
Components of Variation Due to Weight

Source of Variation	D.F	Sum of Squares	Mean Square	F	Test of Hypothesis*
Linear	1	1,735	1,735	17.35	Rejected
Quadratic	1	113	113	1.13	Accepted
Cubic	1	24	24	—	Accepted
Remainder	3	37	12	—	Accepted
Weights	6	1,909			

Regression Equations for the Prediction of "D.L." Values from Weights

Linear Form: $\hat{Y} = 24.8554 - .983921x$,

Quadratic Form: $\hat{Y} = 24.2761 - .983921x + .144829x^2$,

Cubic Form: $\hat{Y} = 24.2761 - .983921x + .144829x^2 - .042056x^3$,

$$\text{where } x = \frac{W - 250}{50}.$$

* The hypothesis tested is a null hypothesis regarding the variation in the same row on the basis of residual term which is indicated in Table 2. For instance, the hypothesis regarding quadratic is that there is no significant quadratic relationship in estimating the "D.L." values from weights. This will always be true throughout the following tables in the same situations.

of variation. The calculation of sums of squares for linear, quadratic, and cubic terms will be illustrated later.

The Main Effects of Rates

The main effect of rates was found to be highly significant, that is, the mean D.L. value varied significantly with the rate ($F = 170.24$, Table 2). The relationship between D.L. values and rates was linear and can be mathematically expressed by the equation

$$\hat{Y} = 24.8554 + 9.5497x, \tag{2}$$

where $x = \dfrac{R - 125}{50}$.

The analysis of variance showed that only the linear effect was significant (Table 4).

TABLE 4
Components of Variation Due to Rate

Source of Variation	D.F.	Sum of Squares	Mean Square	F	Test of Hypothesis
Linear	1	51,071	51,071	510.71	Rejected
Quadratic	1	1	1	—	Accepted
Cubic	1	0	0	—	Accepted
Rates	3	51,072			

Regression Equations for the Prediction of "D.L." Values from Rates

Linear Form: $\hat{Y} = 24.8554 + 9.54978x$,

Quadratic Form: $\hat{Y} = 24.8157 + 9.54978x + .0317x^2$,

Cubic Form: $\hat{Y} = 24.8157 + 9.54978x + .0317x^2 - .00567x^3$,

$$\text{where } x = \frac{R - 125}{50}.$$

The Sex and Rate Combinations

A significant difference was found between the mean D.L. values of the sexes ($F = 315.34$, Table 2).

The equation connecting D.L. values and rate was determined for males as

$$\hat{Y} = 33.2451 + 12.72696x, \text{ where } x = \frac{Rate - 125}{50}. \tag{3}$$

The equation for females was

$$\hat{Y} = 16.4656 + 6.37268x. \tag{4}$$

The components of variation due to sex x rate are given in Table 6. The quadratic component is not significant, so the relation between D.L. values for sex x rate could be adequately represented by the linear equation.

The summary of the results for sex x rate combinations is given in Table 7. The means of the D.L. values for each of the four different rates are given for each sex. The women were more sensitive than the men at each of the four rate levels. The meaning and use of the constants given in columns (8), (9), (10), (11), and (12) and rows (7), (8), and (9) are explained as follows:

An r simply denotes the mean D.L. value of different rates for

TABLE 5

Interaction Between Sex and Rate

Sex		Equations Connecting "D.L." Value and Rate $\hat{Y}=$ "D.L." Value, $x=\frac{\text{Rate}-125}{50}$
Male	Linear:	$\hat{Y}=33.2451+12.72696x$
	Quadratic:	$\hat{Y}=32.8640+12.72696x+.3049x^2$
	Cubic:	$\hat{Y}=32.8640+12.72696x+.3049x^2+.406867x^3$
Female	Linear:	$\hat{Y}=16.4656+\ 6.37268x$
	Quadratic:	$\hat{Y}=16.7675+\ 6.37268x-.2415x^2$
	Cubic:	$\hat{Y}=16.7675+\ 6.37268x-.2415x^2-.408067x^3$

TABLE 6

Components of Variation Due to Sex x Rate

Source of Variation	D.F.	Sum of Squares	Mean Square	F	Test of Hypothesis
Sex x Rate linear	1	5,653	5,653	56.53	Rejected
Sex x Rate quadratic	1	33	33	—	Accepted
Sex x Rate cubic	1	34	34	—	Accepted
Sex x Rate	3	5,720			

each sex and total. B_r, C_r, and D_r denote the linear, the quadratic, and the cubic regression coefficients for each sex and total, respectively. These values can be found in Tables 4 and 5.

$\xi'_1(s)$ denotes the linear polynomial coefficient for each sex;

$\xi'_1(r)$, $\xi'_2(r)$, and $\xi'_3(r)$ denote the linear, the quadratic, and the cubic polynomial coefficients for each rate, respectively. All of the coefficients can be found in Fisher and Yates' Table 22.*†

The sums of squares of various components in Tables 4 and 6 can be calculated from the figures given in Table 7: for instance, to obtain the sums of squares due to the component rate linear (see Table 4), we have (i) to multiply each of the 4 sums of D.L. values

* Fisher, R. A., and Yates, F. *Statistical Tables.* Oliver and Boyd, 1938, pp. 54-59.

†The constants in later tables, i.e., Tables X, XIII, and XVI, have similar meanings and use.

TABLE 7
Summary of Results for Sex x Rate Combinations

Sex	Rate	50	100	150	200	Total	$\xi'_1(s)$	A_r	B_r	C_r	D_r
Male	Sum of Scores	802.9	1508.8	2180.5	2954.7	7446.9	− 1	33.2451	12.72696	.3049	.406867
	Mean	14.3375	26.9429	38.9375	52.7625	33.2451					
Female	Sum of Scores	380.1	736.6	1134.6	1437.0	3688.3	+ 1	16.4656	6.37268	−.2415	−.408067
	Mean	6.7875	13.1536	20.2607	25.6607	16.4656					
Total	Sum of Scores	1183.0	2245.4	3315.1	4391.7	11135.2		24.8554	9.54978	.0317	−.000567
	Mean	10.5625	20.0482	29.5991	39.2116	24.8554					
	$\xi'_1(r)$	− 3	− 1	+ 1	+ 3						
	$\xi'_2(r)$	+ 1	− 1	− 1	+ 1						
	$\xi'_3(r)$	− 1	+ 3	− 3	+ 1						

TABLE 8
Interaction Between Sight and Weight

Sight		Equations Connecting "D.L." Value and Weight $\hat{Y}=$ "D.L." Value, $x=\dfrac{W-250}{50}$
Normal	Linear:	$\hat{Y}=19.7210+.082711x$
	Quadratic:	$\hat{Y}=19.5036+.082711x+.054349x^2$
	Cubic:	$\hat{Y}=19.5036+.082711x+.054349x^2-.030556x^3$
Blind	Linear:	$\hat{Y}=29.9897-2.050564x$
	Quadratic:	$\hat{Y}=29.0485-2.050564x+.2353x^2$
	Cubic:	$\hat{Y}=29.0485-2.050564x+.2353x^2-.053558x^3$

TABLE 9
Components of Variation Due to Sight x Weight

Source of Variation	D.F.	Sum of Squares	Mean Square	F	Test of Hypothesis
Sight x Weight linear	1	2,039	2,039	20.39	Rejected
Sight x Weight quadratic	1	44	44	—	Accepted
Sight x Weight cubic	1	2	2	—	Accepted
Remainder	3	4	1	—	Accepted
	—	——			
Sight x Weight	6	2,089			

in row 5 (or mean values in row 6) by $\xi'_1(r)$, which occurs in the same column as itself; (ii) add up the 4 products; (iii) square the sum; (iv) divide by 112, which is the number of D.L. values for each sex [if in (i) we use the mean values, then we multiply by 112 in this step]; and (v) divide by the sum of squares of $\xi'_1(r)$, namely, 20. Following these steps, we obtain

$$\frac{[1183.0(-3)+2245.4(-1)+3315.1(+1)+4391.7(+3)]^2}{112(20)}=51071$$

or

$$\frac{[10.5625(-3)+20.0482(-1)+29.5991(+1)+39.2116(+3)^2](112)}{20}=51071.$$

For another example, to obtain the sum of squares due to the component sex x rate quadratic, we have (i) to multiply each of the

8 sum of D.L. values (or mean values) by two numbers, namely, values of the $\xi'_1(s)$ and $\xi'_2(r)$, which occur in the same column and in the same row, respectively, as itself; (ii) add up the 8 products; (iii) square the sum; (iv) divide by 56, which is the number of D.L. values for each sex-rate combination [if in (i) we use the mean values, then we multiply by 56 in this step]; and (v) divide by the product of the sum of squares of $\xi'_1(s)$, namely, 2, and of the sum of squares of $\xi'_2(r)$, namely, 4. Following these steps, we obtain

$$\frac{\left[\begin{array}{l}802.9(+1)(-1) + 1508.8(-1)(-1) + 2180.5(-1)(-1) + 2954.7(+1)(-1)\\ + 380.1(+1)(+1) + 736.6(-1)(+1) + 1134.6(-1)(+1) + 1437.0(+1)(+1)\end{array}\right]^2}{56(2)(4)} = 33$$

or

$$\frac{\left[\begin{array}{l}14.3375(+1)(-1) + 26.9429(-1)(-1) + 38.9375(-1)(-1) + 52.7625(+1)(-1)\\ + 6.7875(+1)(+1) + 13.1536(-1)(+1) + 20.2607(-1)(+1) + 25.6607(+1)(+1)\end{array}\right]^2 (56)}{2(4)} = 33.$$

TABLE 10

Summary of Results for Sight x Weight Combinations

Sight	Weight	100	150	200	250	300	350	400	Total	$\xi'_1(i)$	A_w	B_w	C_w	D_w
Normal	Sum of Scores	637.3	616.5	633.4	601.9	644.7	643.4	640.3	4417.5	−1	19.7210	.082711	.054349	−.030556
	Mean	19.9156	19.2656	19.7938	18.8094	20.1469	20.1063	20.0094	19.7210					
Blind	Sum of Scores	1205.4	1074.0	1007.2	917.8	880.4	845.0	787.9	6717.7	+1	29.9897	−2.050564	.235300	−.053558
	Mean	37.6688	33.5625	31.4750	28.6813	27.5125	26.4063	24.6219	29.9897					
Total	Sum of Scores	1842.7	1690.5	1640.6	1519.7	1525.1	1488.4	1428.2	11135.2		24.8554	−.983921	.144829	−.042056
	Mean	28.7922	26.4141	25.6344	23.7453	23.8297	23.2563	22.3156	24.8554					
	$\xi'_1(w)$	−3	−2	−1	0	+1	+2	+3						
	$\xi'_2(w)$	+5	0	−3	−4	−3	0	+5						
	$\xi'_3(w)$	−1	+1	+1	0	−1	−1	+1						

TABLE 11

Interaction Between Sight and Rate

Sight		Equations Connecting "D.L." Value and Rate $\hat{Y}$ = "D.L." Value, $x = \frac{\text{Rate} - 125}{50}$
Normal	Linear:	$\hat{Y} = 19.7210 + 7.66658x$
	Quadratic:	$\hat{Y} = 20.1523 + 7.66658x - .34505x^2$
	Cubic:	$\hat{Y} = 20.1523 + 7.66658x - .34505x^2 - .4074x^3$
Blind	Linear:	$\hat{Y} = 29.9897 + 11.43304x$
	Quadratic:	$\hat{Y} = 29.4794 + 11.43304x + .40845x^2$
	Cubic:	$\hat{Y} = 29.4794 + 11.43304x + .40845x^2 + .4063x^3$

TABLE 12

Components of Variation Due to Sight x Rate

Source of Variation	D.F	Sum of Squares	Mean Square	F	Test of Hypothesis
Sight x Rate linear	1	1,986	1,986	19.86	Rejected
Sight x Rate quadratic	1	64	64	—	Accepted
Sight x Rate cubic	1	33	33	—	Accepted
Sight x Rate	3	2,083			

All the other values of sum of squares in Tables 4 and 6 can be calculated in the same way. Similarly, the values of sum of squares in Table 3 can be calculated in the same way from Table 10.

The Sight and Weight Combinations

It was noted that a highly significant difference in mean D.L. values was found between the normally sighted individuals and those who were congenitally blind. A significant influence was also found between sight x weight (Table 2). The equations between D.L. values and weight were established as follows:

$$\text{For normal sight} \qquad \hat{Y} = 19.7210 + .082711x\,, \tag{5}$$

$$\text{For congenitally blind} \qquad \hat{Y} = 29.9897 - 2.050564x\,, \tag{6}$$

where $x = \frac{W - 250}{50}$ (Table 8).

Only the linear component of the sight x weight interaction was found to be significant (Table 9).

Table 10 summarizes the results for the sight x weight combinations. It is noted that the mean D.L. values for each of the six weights was greater for the blind, that is, the congenitally blind were as a group less sensitive than the normally sighted.

The Sight and Rate Combinations

The equations connecting D.L. values and rate were established as follows for the normally sighted and the congenitally blind, respectively:

$$\text{Normal} \quad \hat{Y} = 19.7210 + 7.66658x\,, \tag{7}$$

$$\text{Blind} \quad \hat{Y} = 29.9897 + 11.43304x\,, \tag{8}$$

where $x = \dfrac{R - 125}{50}$ (Table 11).

The analysis of variance applied to testing the goodness of fit showed that only the linear component of variation due to the sight x rate combination was significant (Table 12).

The summary of the results for the sight x rate combinations is recorded in Table 13. The normally sighted individuals as a group were consistently more sensitive than the congenitally blind as demonstrated by the significantly greater mean D.L. value for the latter at each of the four rates of increasing weights.

The Sex x Sight x Rate Combinations

The sex x sight x rate combination gave the only statistically significant second-order interaction (Table 2). The interaction among sex, sight, and rate shows whether or not the pattern of variation in D.L. values with sex and sight remains the same or not from rate to rate, whether or not the pattern of variation in D.L. values with sight and rate remains the same or not from sex to sex, and whether or not the pattern of variation in D.L. values with sex and rate remains the same or not from the normally sighted to the congenitally blind. The three statements are logically the same and there is one numerical measure of the interaction obtainable from the experimental observations.

We fitted the following equations connecting the D.L. values and rate for each of the four situations (Table 14):

$$\text{Normally sighted males} \quad \hat{Y} = 22.4946 + 8.86216x\,, \tag{9}$$

$$\text{Congenitally blind males} \quad \hat{Y} = 43.9955 + 16.59180x\,, \tag{10}$$

TABLE 13

Summary of Results for Sight x Rate Combinations

Sight	Rate	50	100	150	200	Total	$\xi'_1(i)$	A_r	B_r	C_r	D_r
Normal	Sum of Scores	447.9	888.5	1358.9	1722.2	4417.5	−1	19.7210	7.66658	−.34505	−.407400
	Mean	7.9982	15.8661	24.2661	30.7536	19.7210					
Blind	Sum of Scores	735.1	1356.9	1956.2	2669.5	6717.7	+1	29.9897	11.43304	.40845	.406300
	Mean	13.1268	24.2304	34.9321	47.6696	29.9897					
Total	Sum of Scores	1183.0	2245.4	3315.1	4391.7	11135.2		24.8554	9.54978	.03170	−.000567
	Mean	10.5625	20.0482	29.5991	39.2116	24.8554					
	$\xi'_1(r)$	−3	−1	+1	+3						
	$\xi'_2(r)$	+1	−1	−1	+1						
	$\xi'_3(r)$	−1	+3	−3	+1						

TABLE 14
Interaction Between Sex, Sight, and Rate

Sex	Sight	Equations Connecting "D.L." Value and Rate $\hat{Y}$ = "D.L." Value, $x = \frac{R-125}{50}$	
Male	Normal	Linear:	$\hat{Y} = 22.4946 + 8.86216x$
		Quadratic:	$\hat{Y} = 22.8138 + 8.86216x - .25535x^2$
		Cubic:	$\hat{Y} = 22.8138 + 8.86216x - .25535x^2 + .094033x^3$
	Blind	Linear:	$\hat{Y} = 43.9955 + 16.59180x$
		Quadratic:	$\hat{Y} = 42.9140 + 16.59180x - .86520x^2$
		Cubic:	$\hat{Y} = 42.9140 + 16.59180x - .86520x^2 + .719667x^3$
Female	Normal	Linear:	$\hat{Y} = 16.9473 + 6.47108x$
		Quadratic:	$\hat{Y} = 17.4906 + 6.47108x - .43485x^2$
		Cubic:	$\hat{Y} = 17.4906 + 6.47108x - .43485x^2 - .908900x^3$
	Blind	Linear:	$\hat{Y} = 15.9839 + 6.27430x$
		Quadratic:	$\hat{Y} = 16.0442 + 6.27430x - .04820x^2$
		Cubic:	$\hat{Y} = 16.0442 + 6.27430x - .04820x^2 + .092833x^3$

TABLE 15
Components of Variation Due to Sex x Sight x Rate

Source of Variation	D.F.	Sum of Squares	Mean Square	F	Test of Hypothesis
Sex x Sight x Rate linear	1	2,199	2,199	21.99	Rejected
Sex x Sight x Rate quadratic	1	15	15	—	Accepted
Sex x Sight x Rate cubic	1	2	2	—	Accepted
Sex x Sight x Rate	3	2,216			

Normally sighted females $\quad \hat{Y} = 16.9473 + 6.47108x\,,$ (11)

Congenitally blind females $\quad \hat{Y} = 15.9839 + 6.27430x\,,$ (12)

where $x = \dfrac{R-125}{50}$.

The quadratic and cubic equations were also established but only the linear component of the sex x sight x rate variation was significant (Table 15).

TABLE 16

Summary of Results for Sex x Sight x Rate Combinations

Sex	Sight	Rate	50	100	150	200	Total	$\xi'_1(s)$	$\xi'_1(i)$	A_r	B_r	C_r	D_r
Male	Normal	Sum of Scores	249.7	515.3	758.7	995.7	2519.4	−1	−1	22.4946	8.86216	−.25535	.094033
		Mean	8.9179	18.4036	27.0964	35.5607	22.4946						
	Blind	Sum of Scores	553.2	993.5	1421.8	1959.0	4927.5		+1	43.9955	16.59180	.86520	.719667
		Mean	19.7571	35.4821	50.7786	69.9643	43.9955						
Female	Normal	Sum of Scores	198.2	373.2	600.2	726.5	1898.1	+1	−1	16.9473	6.47108	−.43485	−.908900
		Mean	7.0786	13.3286	21.4357	25.9464	16.9473						
	Blind	Sum of Scores	181.9	363.4	534.4	710.5	1790.2		+1	15.9839	6.27430	−.04820	.092833
		Mean	6.4964	12.9786	19.0857	25.3750	15.9839						
Total		Sum of Scores	1183.0	2245.4	3315.1	4391.7	11135.2			24.8554	9.54978	−.03170	−.000567
		Mean	10.5625	20.0482	29.5991	39.2116	24.8554						
$\xi'_1(r)$			−3	−1	+1	+3							
$\xi'_2(r)$			+1	−1	−1	+1							
$\xi'_3(r)$			−1	+3	−3	+1							

Table 16 summarizes the results presented in this section. In addition, the observed mean D.L. values for each of the four rates are given for the sex-sight combinations. The mean main effects of rate are shown in row (10). The mean over-all rate effects of the sex-sight combinations are given in Column (8). In the case of the men, the normally sighted were uniformly more sensitive than the congenitally blind. On the other hand, the congenitally blind women appeared to be on the average slightly more sensitive to increasing rates than the normally sighted women. The normally sighted women were more sensitive than the normally sighted men. Likewise, the congenitally blind women were more sensitive than the congenitally blind men as they were also than the normally sighted men.

The Summary and Conclusions

The principal purpose of the investigation was to show the application of modern ideas of experimental design to an experiment in psychology. The particular type of design chosen for illustration was the factorial design, a type of design offering much promise in the study of the effects and interactions of psychological factors. The appropriate statistical analysis for this type of design is the analysis of variance, which was applied to the experimental observations. In addition, the mathematical formulation and solution of the problem were carried out; the application of the formulation was made to the psychological problem under consideration (See Part II, following this summary).

The psychological problem studied was that of determining the differential limen of subjects for weights increasing at constant rates. Since there were seven different weights and four different rates of increase, there were twenty-eight different weight-rate combinations. Five trials of each combination were made at each of two experimental periods separated by an interval of one week. The subjects were of both sexes, equal numbers of congenitally blind and of normally sighted individuals. The factorial arrangement was of the type: 4 rates x 7 weights x 2 sexes x 2 sights x 2 dates.

The principal findings were:

1. The adequacy and reliability of the experimental technique were demonstrated by the fact that no appreciable difference was found in the two sets of measurements taken at an interval of one week. The findings from check control stimuli also supplemented this result.
2. From the tests of significance four first-order interactions—sex x sight, sex x rate, sight x weight, sight x rate—and one second-order interaction, sex x sight x rate, were found to be significant. In addition to this direct experimental finding, the finding makes a signifi-

cant contribution to methodology in that it shows that we should first test all interactions before making a complete analysis of variance in any problem.

3. The means of the differential limen (D.L.) values varied systematically with changes in rates. They became greater with the increases in rates.

4. The means of the D.L. values varied systematically with changes in weight. They became less with increases in weight.

5. Significant differences between the means of the D.L. values were found for both sex and sight for the subjects used in this experiment. The women were more sensitive than the men at each of the four rate levels. The normally sighted as a group were more sensitive than the congenitally blind at each of the six weight levels and at each of the four rate levels. The normally sighted men were more sensitive than the congenitally blind men at each of the four rate levels. No marked difference was noticeable between the normally sighted women and the congenitally blind women at any one of the four rate levels.

6. The mathematical relation between D.L. values and each of the factors, and the interacting factors were established. There were twelve regression equations, as follows:

a. D.L. value on weights.
b. D.L. value on rates.
c. D.L. value for each sex on rates.
d. D.L. value for each sight on rates.
e. D.L. value for each sight on weights.
f. D.L. value for each sex and for each sight on rates.

The regression of D.L. values in each case could be graduated by a linear equation. The analysis of variance was used to test goodness of fit of orthogonal polynomial coefficients. With these graduating equations it is possible to compute D.L. values for any particular value of the independent variables within the range of factor levels used in the experiment. The known relations can also be useful in solving problems of estimation.

The methods illustrated in this investigation, and modifications and extensions of them, are capable of very wide application. The general principles can be utilized to various degrees and in a number of ways.

The mathematical statistician is being called upon to develop and analyze statistical designs of increasing complexity since the introduction of the analysis of variance and co-variance. The mathematical formulation and solution of the problem presented in Part I follow.

The reader who is interested in securing a broader background for the mathematical development in Part II is referred to the first comprehensive discussion of the general class of statistical hypotheses, known as "linear" hypothesis, given by St. Kolodziejczyk (4). The bulletin by Jackson (2) and the paper by Johnson and Neyman (3) will also be useful especially in relation to educational and psychological problems. For a review and bibliography of recent developments the paper by Camp (1) is excellent.

PART II: MATHEMATICAL FORMULATIONS

Before we develop the mathematical formulations for our problem in particular, we start with the derivation of those equations necessary in the analysis of variance with only two classifications—say, sex and sight. The numbers in different subclasses are always assumed to be equal. We denote by X_{sit} the D.L. value obtained by the t-th individual in the s-th sex of the i-th sight. The basic assumption in the analysis of variance is that we may write

$$X_{sit} = A + B_s + C_i + I_{si} + z_{sit} , \tag{1}$$

where $s = 1, 2$; $i = 1, 2$; $t = 1, 2, \cdots, n$; 2 denotes the number of sexes and also of sights; and n denotes the number of individuals in each subclass; B_s is a measure of the s-th sex; C_i is a measure of the i-th sight; I_{si} represents the influence of the interaction between sex and sight; z_{sit} represents the error. A is defined as the mean for all groups and individuals; so, furthermore, we assume that

$$\left.\begin{aligned} \sum_s B_s &= 0 \\ \sum_i C_i &= 0 \\ \sum_s \sum_i I_{si} &= 0 \end{aligned}\right\} . \tag{2}$$

To obtain the solution, we first write

$$\chi^2 \sum_s \sum_i \sum_t (X_{sit} - A - B_s - C_i - I_{si})^2 - 2(\lambda_1 \sum_s B_s + \lambda_2 \sum_i C_i + \lambda_3 \sum_s \sum_i I_{si}) , \tag{3}$$

where λ_1, λ_2, and λ_3 are the undetermined multipliers of Lagrange. Minimizing χ^2 with regard to A, B_s, C_i and I_{si}, we obtain*

$$A = \frac{1}{N} \sum_s \sum_i \sum_t X_{sit} = \bar{X}_{\cdots},$$

where $N = 2 \times 2 \times n$,

$$\left.\begin{aligned} B_s &= \frac{1}{2n} \sum_i \sum_t X_{sit} - A - \frac{\sum_i I_{si}}{2} + \frac{\lambda_1}{2n} = \bar{X}_{s\cdot\cdot} - \bar{X}_{\cdots} - \frac{\sum_i I_{si}}{2} + \frac{\lambda_1}{2n} \\ C_i &= \frac{1}{2n} \sum_s \sum_t X_{sit} - A - \frac{\sum_s I_{si}}{2} + \frac{\lambda_2}{2n} = \bar{X}_{\cdot i \cdot} - \bar{X}_{\cdots} - \frac{\sum_s I_{si}}{2} + \frac{\lambda_2}{2n} \\ I_{si} &= \frac{1}{n} \sum_t X_{sit} - A - B - C_i + \frac{\lambda_3}{n}. \end{aligned}\right\} \quad (4)$$

From equations (2) and (4), we get

$$\sum_s B_s = \sum_s \bar{X}_{s\cdot\cdot} - 2\bar{X}_{\cdots} - \frac{\sum_s \sum_i I_{si}}{2} + \frac{\lambda_1}{n}, \quad (5)$$

which reduces to

$$\lambda_1 = 0. \quad (6)$$

Similarly

$$\lambda_2 = \lambda_3 = 0. \quad (7)$$

By the method of elimination, we get

$$\chi^2_q = \sum_s \sum_i \sum_t (X_{sit} - \bar{X}_{si\cdot})^2 = \sum_s \sum_i \sum_t X^2_{sit} - \sum_s \sum_i \{n \bar{X}^2_{si\cdot}\}. \quad (8)$$

The hypothesis we wish to test first is

$$H_0 : I_{si} = 0, \quad (9)$$

i.e., the hypothesis that there is no influence of the interaction between sex and sight. Assuming that H_0 is true, we have, from equation (3),

$$\chi^2_{r_0} = \sum_s \sum_i \sum_t (X_{sit} - A - B_s - C_i)^2 - 2(\alpha_1 \sum_s B_s + \alpha_2 \sum_i C_i), \quad (10)$$

where α_1 and α_2 are the undetermined multipliers of Lagrange.

* A dot in place of subscript is used to indicate that the variable has been averaged with respect to that subscript. This will always be true throughout this paper.

Minimizing $\chi^2_{r_0}$ with regard to A, B_s, and C_i, we obtain

$$\left.\begin{aligned} A &= \bar{X}_{...} \\ B_s &= \bar{X}_{s..} + \bar{X}_{...} + \frac{\alpha_1}{2n} \\ C_i &= \bar{X}_{.i.} - \bar{X}_{...} + \frac{\alpha_2}{2n} \end{aligned}\right\} , \tag{11}$$

where

$$\alpha_1 = \alpha_2 = 0 . \tag{12}$$

Substituting these values in equation (10) and simplifying, we obtain the relative minimum value $\chi^2_{r_0}$:

$$\begin{aligned} \chi^2_{r_0} &= \sum_s \sum_i \sum_t (X_{sit} - \bar{X}_{s..} - \bar{X}_{.i.} + \bar{X}_{...})^2 \\ &= \chi^2_a + \sum_s \sum_i \sum_t (\bar{X}_{si.} - \bar{X}_{s..} - \bar{X}_{.i.} + \bar{X}_{...})^2 \\ &= \chi^2_a + \sum_s \sum_i \{n\, \bar{X}^2_{si.}\} - \sum_s \{2\, n\, \bar{X}^2_{s..}\} - \sum_i \{2\, n\, \bar{X}^2_{.i.}\} + N\, \bar{X}^2_{...} \\ &= \chi^2_a + \chi^2_0 . \end{aligned} \tag{13}$$

Then we may test the relative hypothesis on the basis of $\chi^2_{r_0}$:

$$H_1 : B_s = 0 , \tag{14}$$

i.e., the hypothesis that there is no significant difference between sex. Assuming that H_1 is true, we may write

$$\chi^2_{r_1} = \sum_s \sum_i \sum_t (X_{sit} - A - C_i)^2 - 2\, \beta \sum_i C_i , \tag{15}$$

where β is the undetermined multiplier of Lagrange. Minimizing $\chi^2_{r_1}$ with regard to A and C_i, we have

$$\left.\begin{aligned} A &= \bar{X}_{...} \\ C_i &= \bar{X}_{.i.} - \bar{X}_{...} + \frac{\beta}{2n} \end{aligned}\right\} , \tag{16}$$

where

$$\beta = 0 . \tag{17}$$

Substituting all the values into the equation (15) we obtain

$$\begin{aligned} \chi^2_{r_1} &= \sum_s \sum_i \sum_t (X_{sit} - \bar{X}_{.i.})^2 = \chi^2_a + \chi^2_0 + \sum_s \{2\, n\, \bar{X}^2_{s..}\} - N \bar{X}^2_{...} \\ &= \chi^2_a + \chi^2_0 + \chi^2_1 . \end{aligned} \tag{18}$$

Finally, we may test the relative hypothesis on the basis of $\chi^2_{r_1}$:

$$H_2 : C_i = 0\,, \tag{19}$$

i.e., the hypothesis that there is no significant difference between sights.

Assuming that H_2 is true and proceeding as before, we obtain

$$\chi^2_{r_2} = \sum_s \sum_i \sum_t (X_{sit} - \bar{X}_{...})^2 = \sum_s \sum_i \sum_t X^2_{sit} - N\bar{X}^2_{...}$$

$$= \chi^2_a + \chi^2_0 + \chi^2_1 + \sum_i \{2\,n\,\bar{X}^2_{.i.}\} - N\bar{X}^2_{...} = \chi^2_a + \chi^2_0 + \chi^2_1 + \chi^2_2. \tag{20}$$

From the equation (20), the additive property of the sum of squares is readily demonstrated. It is also noted, in the case of equal numbers in subclasses, that there is only one answer for each hypothe-

TABLE 1

Analysis of Variance of D. L. Values of Different Sexes and Different Sights

Source of Variations	d.f.	Sum of Squares*
Error	$N-4$	$\sum_s \sum_i \sum_t X^2_{sit} - \sum_s \sum_i \{n\,\bar{X}^2_{si}\}$
Sex x Sight	1	$\sum_s \sum_i \{n\,\bar{X}^2_{si.}\} - \sum_s \{2\,n\,\bar{X}^2_{s..}\} - \sum_i \{2\,n\,\bar{X}^2_{.i.}\} + N\bar{X}^2_{...}$
Sex	1	$\sum_s \{2\,n\,\bar{X}^2_{s..}\} - N\bar{X}^2_{...}$
Sight	1	$\sum_i \{2\,n\,\bar{X}^2_{.i.}\} - N\bar{X}^2_{...}$
Total	$N-1$	$\sum_s \sum_i \sum_t X^2_{sit} - N\bar{X}^2_{...}$

* Mathematically speaking,

$$\frac{(\Sigma X)^2}{N} \equiv N\bar{X}^2.$$

So we have written $N\,\bar{X}^2_{...}$ instead of $\left(\dfrac{\sum_s \sum_i \sum_t X_{sit}}{N}\right)^2$,

$\sum_s \{2\,n\,\bar{X}^2_{s..}\}$ instead of $\sum_s \left\{\dfrac{(\sum_i \sum_t X_{sit})^2}{2\,n}\right\}$,

and so on. However, in doing calculating work, $\sum_s \left\{\dfrac{(\sum_i \sum_t X_{sit})^2}{2\,n}\right\}$ is more accurate than $\sum_s \{2\,n\,\bar{X}^2_{..}\}$ from the viewpoint of significant figures. We use the latter in the presentation of the formulas only for the sake of simplicity. This holds true throughout the present article.

sis tested whatever the order of testing is. This point should always be kept in mind throughout the present paper. All the results obtained may be summarized in Table 1.

Now we shall work out the equations with three classifications—say, sex, sight and standard weights. Denote by X_{sijt} the D.L. value on the j-th weight obtained by the t-th individual of the s-th sex of the i-th sight.

The basic assumption in the analysis of variance is that we may write

$$X_{sijt} = A + B_s + C_i + D_j + I_{si} + I_{sj} + I_{ij} + I_{sij} + z_{sijt}\,, \qquad (21)$$

where $s = 1, 2;\ i = 1, 2;\ j = 1, 2, \cdots 7;\ t = 1, 2, \cdots, n$; 2 denotes the number of sexes and also of sights; 7 denotes the number of weights; and n denotes the number of individuals in each subclass; B_s is a measure of the s-th sex; C_i is a measure of the i-th sight; D_j is a measure of the j-th weight; I_{si} represents the influence of the interaction bewteen sex and sight; I_{sj}, sex and weight; I_{ij}, sight and weight; I_{sij}, sex, sight, and weight; z_{sijt}, the error. A is defined as the mean for all groups and individuals. Furthermore, we assume that

$$\left.\begin{array}{ll} \sum\limits_s B_s = 0, & \sum\limits_s \sum\limits_j I_{sj} = 0 \\ \sum\limits_i C_i = 0, & \sum\limits_i \sum\limits_j I_{sj} = 0 \\ \sum\limits_j D_j = 0, & \sum\limits_s \sum\limits_i \sum\limits_j I_{sij} = 0 \\ \sum\limits_s \sum\limits_i I_{si} = 0, & \end{array}\right\} . \qquad (22)$$

To obtain the absolute minimum, we write

$$\begin{aligned} \chi^2 = & \sum_s \sum_i \sum_j \sum_t (X_{sijt} - A - B_s - C_i - D_j - I_{si} - I_{sj} - I_{ij} - I_{sij})^2 \\ & - 2(\lambda_1 \sum_s B_s + \lambda_2 \sum_i C_i + \lambda_3 \sum_j D_j + \lambda_4 \sum_s \sum_i I_{si} + \lambda_5 \sum_s \sum_j I_{si} \\ & + \lambda_6 \sum_i \sum_j I_{ij} + \lambda_7 \sum_s \sum_i \sum_j I_{sij})\,, \end{aligned} \qquad (23)$$

where $\lambda_1, \lambda_2, \cdots, \lambda_7$ are the undetermined multipliers of Lagrange.

Minimizing χ^2 with regard to A, B_s, C_i, D_j, I_{si}, I_{sj}, I_{ij} and I_{sij}, we have

$$\left.\begin{aligned}
A &= \bar{X}_{....} \\
B_s &= \bar{X}_{s...} - \bar{X}_{....} - \frac{\sum_i I_{si}}{2} - \frac{\sum_j I_{sj}}{7} - \frac{\sum_i\sum_j I_{sij}}{14} + \frac{\lambda_1}{14n} \\
C_i &= \bar{X}_{.i..} - \bar{X}_{....} - \frac{\sum_s I_{si}}{2} - \frac{\sum_j I_{ij}}{7} - \frac{\sum_s\sum_j I_{sij}}{14} + \frac{\lambda_2}{14n} \\
D_j &= \bar{X}_{..j.} - \bar{X}_{....} - \frac{\sum_s I_{sj}}{2} - \frac{\sum_i I_{ij}}{2} - \frac{\sum_s\sum_i I_{sij}}{4} + \frac{\lambda_3}{4n} \\
I_{si} &= \bar{X}_{si..} - \bar{X}_{....} - B_s - C_i - \frac{\sum_j I_{sj}}{7} - \frac{\sum_j I_{ij}}{7} - \frac{\sum_j I_{sij}}{7} + \frac{\lambda_4}{7n} \\
I_{sj} &= \bar{X}_{s.j.} - \bar{X}_{....} - B_s - D_j - \frac{\sum_i I_{si}}{2} - \frac{\sum_i I_{ij}}{2} - \frac{\sum_i I_{sij}}{2} + \frac{\lambda_5}{2n} \\
I_{ij} &= \bar{X}_{.ij.} - \bar{X}_{....} - C_i - D_j - \frac{\sum_s I_{si}}{2} - \frac{\sum_i I_{ij}}{2} - \frac{\sum_s I_{sij}}{2} + \frac{\lambda_6}{2n} \\
I_{sij} &= \bar{X}_{sij} - \bar{X}_{....} - B_s - C_i - D_j - I_{si} - I_{sj} - I_{ij} + \frac{\lambda_7}{n}
\end{aligned}\right\} , \quad (24)$$

where

$$\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_7 = 0\,. \tag{25}$$

By the method of elimination, we obtain

$$\chi^2_a = \sum_s\sum_i\sum_j\sum_t (X_{sijt} - \bar{X}_{sij})^2 = \sum_s\sum_i\sum_j\sum_t X^2_{sijt} - \sum_s\sum_i\sum_j \{n\,\bar{X}^2_{sij}\}. \tag{26}$$

The hypothesis we wish to test first is

$$H_{0_1} : I_{sij} = 0\,, \tag{27}$$

i.e., the hypothesis that there is no significant influence of the interaction between sex, sight, and weight. Assuming that H_{0_1} is true, we have, from equation (26),

$$\chi^2 r_{0_3} = \sum_s \sum_i \sum_j \sum_t (X_{sijt} - A - B_s - C_i - D_j - I_{si} - I_{sj} - I_{ij})^2$$
$$- 2(\alpha_1 \sum_s B_s + \alpha_2 \sum_i C_i + \alpha_3 \sum_j D_j + \alpha_4 \sum_s \sum_i I_{si} + \alpha_5 \sum_s \sum_j I_{sj}$$
$$+ \alpha_6 \sum_i \sum_j I_{ij}), \tag{28}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_6$ are the undetermined multipliers of Lagrange. Minimizing $\chi^2 r_{0_3}$ with regard to A, B_s, C_i, D_j, I_{si}, I_{sj}, and I_{ij}, we have

$$\left.\begin{aligned}
A &= \bar{X}_{....} \\
B_s &= \bar{X}_{s...} - \bar{X}_{....} - \frac{\sum_i I_{si}}{2} - \frac{\sum_j I_{sj}}{2} + \frac{\alpha_1}{14n} \\
C_i &= \bar{X}_{.i..} - \bar{X}_{....} - \frac{\sum_s I_{si}}{2} - \frac{\sum_j I_{ij}}{7} + \frac{\alpha_2}{14n} \\
D_j &= \bar{X}_{..j.} - \bar{X}_{....} - \frac{\sum_s I_{sj}}{2} - \frac{\sum_i I_{ij}}{2} + \frac{\alpha_3}{2n} \\
I_{si} &= \bar{X}_{si..} - \bar{X}_{....} - B_s - C_i - \frac{\sum_j I_{sj}}{7} - \frac{\sum_j I_{ij}}{7} + \frac{\alpha_4}{7n} \\
I_{sj} &= \bar{X}_{s.j.} - \bar{X}_{....} - B_s - D_j - \frac{\sum_i I_{si}}{2} - \frac{\sum_i I_{ij}}{2} + \frac{\alpha_5}{2n} \\
I_{ij} &= \bar{X}_{.ij.} - \bar{X}_{....} - C_i - D_j - \frac{\sum_s I_{si}}{2} - \frac{\sum_s I_{sj}}{2} + \frac{\alpha_6}{2n}
\end{aligned}\right\}, \tag{29}$$

where

$$\alpha_1 = \alpha_2 = \cdots = \alpha_6 = 0 . \tag{30}$$

By the method of elimination, we have

$$\chi^2_{r_{0_1}} = \sum_s \sum_i \sum_j \sum_t (X_{sijt} - \bar{X}_{si..} - \bar{X}_{s.j.} - \bar{X}_{.ij.} + \bar{X}_{s...} + \bar{X}_{.j..} + \bar{X}_{..j.} - \bar{X}_{....})^2$$

$$= \chi^2_a + \sum_s \sum_i \sum_j \{n\,\bar{X}^2_{sij.}\} - \sum_s \sum_i \{7\,n\,\bar{X}^2_{si..}\} - \sum_s \sum_j \{2\,n\,\bar{X}^2_{s.j.}\}$$

$$- \sum_i \sum_j \{2\,n\,\bar{X}^2_{.ij.}\} + \sum_s \{14\,n\,\bar{X}^2_{s...}\} + \sum_i \{14\,n\,\bar{X}^2_{.j..}\} + \sum_j \{4\,n\,\bar{X}^2_{..j.}\} - N\,\bar{X}^2_{....}$$

$$= \chi^2_a + \chi^2_0 \,. \tag{31}$$

Then we may test the following relative hypothesis on the basis of $\chi^2_{r_{0_1}}$:

$$H_{0_2}\colon I_{si} = 0\,, \tag{32}$$

i.e., the hypothesis that there is no significant influence of interaction between sex and sight. Assuming that H_{0_2} is true, we may write

$$\begin{aligned}\chi^2_{r_{0_2}} &= \sum_s \sum_i \sum_j \sum_t (X_{sijt} - A - B_s - C_i - D_j - I_{sj} - I_{ij})^2 \\ &\quad - 2(\beta_1 \sum_s B_s + \beta_2 \sum_i C_i + \beta_3 \sum_j D_j + \beta_4 \sum_s \sum_j I_{sj} + \beta_5 \sum_i \sum_j I_{ij})\,;\end{aligned} \tag{33}$$

where $\beta_1, \beta_2, \cdots, \beta_5$ are the undetermined multipliers of Lagrange. Minimizing $\chi^2_{r_{0_2}}$ with regard to A, B_s, C_i, D_j, I_{sj}, and I_{ij}, we have

$$\left.\begin{aligned}
A &= \bar{X}_{....} \\
B_s &= \bar{X}_{s...} - \bar{X}_{....} - \frac{\sum_j I_{sj}}{7} + \frac{\beta_1}{14\,n} \\
C_i &= \bar{X}_{.j..} - \bar{X}_{....} - \frac{\sum_j I_{ij}}{7} + \frac{\beta_2}{14\,n} \\
D &= \bar{X}_{..j.} - \bar{X}_{....} - \frac{\sum_s I_{sj}}{2} - \frac{\sum_i I_{ij}}{2} + \frac{\beta_3}{4\,n} \\
I_{sj} &= \bar{X} \quad - \bar{X}_{....} - B \ - D_j - \frac{\sum_i I_{ij}}{2} + \frac{\beta_4}{2\,n} \\
I_{ij} &= \bar{X}_{.ij.} - \bar{X}_{....} - C_i - D_j - \frac{\sum_s I_{sj}}{2} + \frac{\beta_5}{2\,n}
\end{aligned}\right\} , \tag{34}$$

where

$$\beta_1 = \beta_2 = \cdots \beta_5 = 0 \,. \tag{35}$$

By the method of elimination, we have

$$\begin{aligned}\chi^2_{r_{0_2}} &= \sum_s \sum_i \sum_j \sum_t (X_{sijt} - \bar{X}_{s.j.} - \bar{X}_{.ij.} + \bar{X}_{..j.} - 2\,\bar{X}_{....})^2 \\ &= \chi^2_a + \chi^2_{0_1} + \sum_s \sum_j \{7\,n\,\bar{X}^2_{si..}\} - \sum_s \{14n\,\bar{X}^2_{s...}\} \\ &\qquad - \sum_i \{14\,n\,\bar{X}^2_{.j..}\} + N\,\bar{X}^2_{....} \\ &= \chi^2_a + \chi^2_{0_1} + \chi^2_{0_2} \,.\end{aligned} \tag{36}$$

Next we wish to test the following relative hypothesis on the basis of $\chi^2_{r_{0_2}}$:

$$H_{0_3} : I_{sj} = 0 \tag{37}$$

i.e., the hypothesis that there is no significant influence of interaction between sex and weight. Assuming that H_{0_3} is true, we may write

$$\begin{aligned}\chi^2_{r_{0_3}} &= \sum_s \sum_i \sum_j \sum_t (X_{sijt} - A - B_s - C_i - D_j - I_{ij})^2 \\ &\quad - 2(\gamma_1 \sum_s B_s + \gamma_2 \sum_i C_i + \gamma_3 \sum_j D_j + \gamma_4 \sum_i \sum_j I_{ij}) \,,\end{aligned} \tag{38}$$

where γ_1, γ_2, γ_3, and γ_4 are the undetermined multipliers of Lagrange. Proceeding as before, we obtain

$$\begin{aligned}\chi^2_{r_{0_3}} &= \sum_s \sum_i \sum_j \sum_t (X_{sijt} - \bar{X}_{.ij.} - \bar{X}_{s...} + \bar{X}_{....})^2 \\ &= \chi^2_a + \chi^2_{0_1} + \chi^2_{0_2} + \sum_s \sum_j \{2\,n\,\bar{X}^2_{s.j.}\} - \sum_s \{14\,n\,\bar{X}^2_{s...}\} \\ &\qquad - \sum_j \{4\,n\,\bar{X}^2_{..j.}\} + N\,\bar{X}^2_{....} \\ &= \chi^2_a + \chi^2_{0_1} + \chi^2_{0_2} + \chi^2_{0_3} \,.\end{aligned} \tag{39}$$

Again, we wish to test the following relative hypothesis on the basis of $\chi^2_{r_{0_3}}$:

$$H_{0_4} : I_{ij} = 0 \,, \tag{40}$$

i.e., the hypothesis that there is no significant influence of interaction between sight and weight. Assuming that H_{0_4} is true, we have

$$\begin{aligned}\chi^2_{r_{0_4}} &= \sum_s \sum_i \sum_j \sum_t (X_{sijt} - A - B_s - C_i - D_j)^2 \\ &\quad - 2(\delta_1 \sum_s B_s + \delta_2 \sum_i C_i + \delta_3 \sum_j D_j) \,,\end{aligned} \tag{41}$$

where δ_1, δ_2, and δ_3 are the undetermined multipliers of Lagrange. Proceeding as before, we get

$$\begin{aligned}\chi^2_{r_{0_4}} &= \sum_s \sum_i \sum_j \sum_t (X_{sijt} - \overline{X}_{s\ldots} - \overline{X}_{.j..} - \overline{X}_{..j.} + 2\,\overline{X}_{\ldots.})^2 \\ &= \chi^2_a + \chi^2_{0_1} + \chi^2_{0_2} + \chi^2_{0_3} + \sum_i \sum_j \{2\,n\,\overline{X}^2_{.ij.}\} \\ &\qquad - \sum_i \{14\,n\,\overline{X}^2_{.j..}\} - \sum_j \{4\,n\,\overline{X}^2_{..j.}\} + N\,\overline{X}^2_{\ldots.} \\ &= \chi^2_a + \chi^2_{0_1} + \chi^2_{0_2} + \chi^2_{0_3} + \chi^2_{0_4}\,. \end{aligned} \tag{42}$$

After we have tested all the hypotheses dealing with interactions either of first order or of second order, we come to the tests of main effects. The first hypothesis we wish to test is

$$H_1 : B_s = 0\,, \tag{43}$$

i.e., the hypothesis that there is no significant difference between sexes. Assuming that this hypothesis is true, we may write

$$\begin{aligned}\chi^2_{r_1} &= \sum_s \sum_i \sum_j \sum_t (X_{sijt} - A - C_i - D_j)^2 \\ &\qquad - 2(\varepsilon_1 \sum_i C_i + \varepsilon_2 \sum_j D_j)\,, \end{aligned} \tag{44}$$

where ε_1 and ε_2 are undetermined multipliers of Lagrange. Proceeding as before, we have

$$\begin{aligned}\chi^2_{r_1} &= \sum_s \sum_i \sum_j \sum_t (X_{sijt} - \overline{X}_{.j..} - \overline{X}_{..j.} + \overline{X}_{\ldots.})^2 \\ &= \chi^2_a + \chi^2_{0_1} + \chi^2_{0_2} + \chi^2_{0_3} = \chi^2_{0_4} + \sum_s \{14\,n\,\overline{X}^2_{s\ldots}\} - N\,\overline{X}^2_{\ldots.} \\ &= \chi^2_a + \chi^2_{0_1} + \chi^2_{0_2} + \chi^2_{0_3} + \chi^2_{0_4} + \chi^2_1\,. \end{aligned} \tag{45}$$

The next hypothesis is

$$H_2 : C_i = 0\,, \tag{46}$$

i.e., the hypothesis that there is no significant difference between sights. Assuming H_2 is true, we may write

$$\chi^2_{r_2} = \sum_s \sum_i \sum_j \sum_t (X_{sijt} - A - D_j)^2 - 2\,\rho \sum_j D_j\,, \tag{47}$$

where ρ is the undetermined multiplier of Lagrange. By using the same method as before, we obtain

$$\chi^2_{r_2}=\sum_s\sum_i\sum_j\sum_t(X_{sijt}-\bar{X}_{..j.})^2=\chi^2_a+\chi^2_{0_1}+\chi^2_{0_2}$$
$$+\chi^2_{0_3}+\chi^2_{0_4}+\chi^2_1+\sum_i\{14\,n\,\bar{X}^2_{.i..}\}-N\,\bar{X}^2_{....}$$
$$=\chi^2_a+\chi^2_{0_1}+\chi^2_{0_2}+\chi^2_{0_3}+\chi^2_{0_4}+\chi^2_1+\chi^2_2. \tag{48}$$

Finally, the following hypothesis should be tested:

$$H_3 : D_j = 0\,, \tag{49}$$

i.e., the hypothesis that there is no significant difference between weights. Assuming that H_3 is true, we may write

$$\chi^2_{r_3}=\sum_s\sum_i\sum_j\sum_t(X_{sijt}-A)^2, \tag{50}$$

which reduces easily to

$$\chi^2_{r_3}=\sum_s\sum_i\sum_j\sum_t(X_{sijt}-\bar{X}_{....})^2=\sum_s\sum_i\sum_j\sum_t X^2_{sijt}-N\,\bar{X}^2_{....}$$
$$=\chi^2_a+\chi^2_{0_1}+\chi^2_{0_2}+\chi^2_{0_3}$$
$$+\chi^2_{0_4}+\chi^2_1+\chi^2_2+\sum_j\{4\,n\,\bar{X}^2_{..j.}\}-N\,\bar{X}^2_{....}$$
$$=\chi^2_a+\chi^2_{0_1}+\chi^2_{0_2}+\chi^2_{0_3}+\chi^2_{0_4}+\chi^2_1+\chi^2_2+\chi^2_3. \tag{51}$$

Again, the additive property of the sum of squares is clearly demonstrated in equation (51). All the results may be summarized in Table 2.

Now we come to our own problem which includes five variables:

2 sexes x 2 sights x 7 weights x 4 rates x 2 dates .

The mathematical expression of the D.L. value made by t-th individual in s-th sex of i-th sight on j-th weight of k-th rate at l-th date is as follows:

$$X_{sijklt}=A+B_s+C_i+D_j+E_k+F_l+I_{si}+I_{sj}+I_{sk}+I_{sl}$$
$$+I_{ij}+I_{ik}+I_{il}+I_{jl}+I_{jk}+I_{jl}+I_{sij}+I_{sik}+I_{sil}+I_{sjk}$$
$$+I_{sjl}+I_{skl}+I_{ijk}+I_{ijl}+I_{ikl}+I_{jkl}+I_{sijk}+I_{sijl}$$
$$+I_{sikl}+I_{sjkl}+I_{ijkl}+I_{sikjl}+z_{sijklt}\,, \tag{52}$$

where the subscripts s, i, j, k, l, and t refer to sex, sight, weight, rate, date, and the particular individual, respectively; A is the grand mean

TABLE 2

Analysis of Variance of D.L. Values on Different Weights of Different Sex and Different Sights

Source of Variation	d.f.	Sum of Squares*
Error	$N-28$	$a-b$
Sex x sight x weight	6	$b-c_1-c_2-c_3+d_1+d_2+d_3-e$
Sex x sight	1	$c_1-d_1-d_2+e$
Sex x weight	6	$c_2-d_1-d_3+e$
Sight x weight	6	$c_3-d_2-d_3+e$
Sex	1	d_1-e
Sight	1	d_2-e
Weight	6	d_3-e
Total	$N-1$	$a-e$

* where

$$a = \sum_s \sum_i \sum_j \sum_t X^2_{sijt} \quad ;$$

$$b_1 = \sum_s \sum_i \sum_j \{n \bar{X}^2_{sij\cdot}\} \quad ;$$

$$c_1 = \sum_s \sum_i \{7 n \bar{X}^2_{si\cdot\cdot}\}, \quad c_2 = \sum_s \sum_j \{2 n \bar{X}^2_{s\cdot j\cdot}\}, \quad c_3 = \sum_i \sum_j \{2 n \bar{X}^2_{\cdot ij\cdot}\} \quad ;$$

$$d_1 = \sum_s \{14 \bar{X}^2_{s\cdots}\}, \quad d_2 = \sum_i \{14 n \bar{X}^2_{\cdot i\cdot\cdot}\}, \quad d_3 = \sum_j \{4 n \bar{X}^2_{\cdot\cdot j\cdot}\} \quad ;$$

$$e = N \bar{X}^2_{\cdots\cdot} \, .$$

of all individuals; B, C, D, E, and F are the measures of the main effects with respect to their own subscripts; I's are the measures of the interactions with respect to their own subscripts: z_{sijklt} is the error. So we have*

1 grand mean;
5 measures of main effects;
10 measures of interactions of first order;
10 measures of interactions of second order;
5 measures of interactions of third order;
1 measure of interaction of fourth order.

Obviously, the numbers of the categories are similar to the coefficients involved in the expansion of $(a+b)^5$. If we go back to the earlier developments regarding two and three variables, we have the same demonstration. For two variables, we have

1 grand mean;
2 measures of main effects;
1 measure of interaction of first order.

* First order refers to the interaction between two variables; second order refers to the interaction between three variables; and so on.

For three variables, we have

1 grand mean;
3 measures of main effect;
3 measures of interactions of first order;
1 measure of interaction of second order.

Hence, for n variables, we have

1 grand mean;

n measures of main effects;

$\frac{n(n-1)}{2!}$ measures of interactions of first order;

$\frac{n(n-1)(n-2)}{3!}$ measures of interactions of second order;

. . . .

$\frac{n(n-1)(n-2)}{3!}$ measures of interaction of $(n-4)$-th order;

$\frac{n(n-1)}{2!}$ measures of interactions of $(n-3)$-th order;

n measures of interactions of $(n-2)$-th order;

1 measure of interaction of $(n-1)$-th order.

Before we develop the equations necessary to our problem in particular, we should always keep in mind that there are so many measures of interactions which should be taken into consideration. Usually the higher orders of interaction are not significant, so some students insist that those higher orders of interaction might be included in the error variance. However, it may be found that some of the higher orders of interaction are also significant and that the lower orders of interaction are not all significant. Therefore, we should first test every kind of interaction on the basis of error. Then we may put all the insignificant interactions in error term. This point has been observed completely in the analysis of our problem.

The mathematical formulations for our problem can be obtained by following the same method as used before. In order to conserve space we simply summarize all the results in Table 3, in which we assume,

$$a = \sum_{s}\sum_{i}\sum_{j}\sum_{k}\sum_{l}\sum_{t} X^2_{sijklt};$$

$$b = \sum_{s}\sum_{i}\sum_{j}\sum_{k}\sum_{l}\{2\,\bar{X}^2_{sijkl\cdot}\};$$

$$c_1 = \sum_{s}\sum_{i}\sum_{j}\sum_{k}\{4\,\bar{X}^2_{sijk\cdot\cdot}\}, \quad c_2 = \sum_{s}\sum_{i}\sum_{j}\sum_{l}\{8\,\bar{X}^2_{sij\cdot l\cdot}\}, \quad c_3 = \sum_{s}\sum_{i}\sum_{k}\sum_{l}\{14\,\bar{X}^2_{si\cdot kl\cdot}\},$$

$$c_4 = \sum_{s}\sum_{j}\sum_{k}\sum_{l}\{4\,\bar{X}^2_{s\cdot jkl}\}, \quad c_5 = \sum_{i}\sum_{j}\sum_{k}\sum_{l}\{2\,\bar{X}^2_{\cdot ijkl\cdot}\};$$

$$d_1 = \sum_{s}\sum_{i}\sum_{j}\{16\,\bar{X}^2_{sij\cdots}\}, \quad d_2 = \sum_{s}\sum_{i}\sum_{k}\{28\,\bar{X}^2_{si\cdot k\cdot\cdot}\}, \quad d_3 = \sum_{s}\sum_{i}\sum_{l}\{56\,\bar{X}^2_{si\cdot\cdot l\cdot}\},$$

$$d = \sum_{s}\sum_{j}\sum_{k}\{8\,\bar{X}^2_{s\cdot jk\cdot\cdot}\}, \quad d_5 = \sum_{s}\sum_{j}\sum_{l}\{16\,\bar{X}^2_{s\cdot j\cdot l\cdot}\}, \quad d_6 = \sum_{s}\sum_{k}\sum_{l}\{28\,\bar{X}^2_{s\cdot\cdot kl\cdot}\},$$

$$d_7 = \sum_{i}\sum_{j}\sum_{k}\{8\,\bar{X}^2_{\cdot ijk\cdot\cdot}\}, \quad d_8 = \sum_{i}\sum_{j}\sum_{l}\{16\bar{X}^2_{\cdot ij\cdot l\cdot}\}, \quad d_9 = \sum_{i}\sum_{k}\sum_{l}\{28\,\bar{X}^2_{\cdot i\cdot kl\cdot}\},$$

$$d_{10} = \sum_{j}\sum_{k}\sum_{l}\{8\,\bar{X}^2_{\cdot\cdot jkl\cdot}\};$$

$$e_1 = \sum_{s}\sum_{i}\{112\,\bar{X}^2_{si\cdots\cdot}\}, \quad e_2 = \sum_{s}\sum_{j}\{32\,\bar{X}^2_{s\cdot j\cdots}\}, \quad e_3 = \sum_{s}\sum_{k}\{56\,\bar{X}^2_{s\cdot\cdot k\cdot\cdot}\},$$

$$e_4 = \sum_{s}\sum_{l}\{112\,\bar{X}^2_{s\cdots\cdot l}\}, \quad e_5 = \sum_{i}\sum_{j}\{32\,\bar{X}^2_{\cdot ij\cdots}\}, \quad e_6 = \sum_{i}\sum_{k}\{56\,\bar{X}^2_{\cdot i\cdot k\cdot\cdot}\},$$

$$e_7 = \sum_{i}\sum_{l}\{112\,\bar{X}^2_{\cdot i\cdot\cdot l\cdot}\}, \quad e_8 = \sum_{j}\sum_{k}\{16\,\bar{X}^2_{\cdot\cdot jk\cdot\cdot}\}, \quad e_9 = \sum_{j}\sum_{l}\{32\,\bar{X}^2_{\cdot\cdot j\cdot l\cdot}\},$$

$$e_{10} = \sum_{k}\sum_{1}\{56\,\bar{X}^2_{\cdots k1\cdot}\};$$

$$f_1 = \sum_{s}\{224\,\bar{X}^2_{s\cdots\cdots}\}, \quad f_2 = \sum_{i}\{224\,\bar{X}^2_{\cdot i\cdots\cdot}\}, \quad f_3 = \sum_{j}\{64\,\bar{X}^2_{\cdot\cdot j\cdots}\},$$

$$f_4 = \sum_{k}\{112\,\bar{X}^2_{\cdots k\cdot\cdot}\}, \quad f_5 = \sum_{l}\{224\,\bar{X}^2_{\cdots\cdot l\cdot}\};$$

$$g = 448\,\bar{X}^2_{\cdots\cdots}\,.$$

TABLE 3

Analysis of Variance of D.L. Values for the Factorial Difference 4 x 7 x 2 x 2 x 2

Source of Variation	d.f.	Sum of Squares
Error	224	$a - b$
Interaction 4th order Sex x sight x weight x rate x date	18	$b - \sum_{p=1}^{5} c_p + \sum_{p=1}^{10} d_p - \sum_{p=1}^{10} e_p + \sum_{p=1}^{5} f_p - g$
3rd order Sex x sight x weight x rate	18	$c_1 - (d_1 + d_2 + d_4 + d_7) + (e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_8) - (f_1 + f_2 + f_3 + f_4) + g$
Sex x sight x weight x date	6	$c_2 - (d_1 + d_3 + d_5 + d_8) + (e_1 + e_2 + e_4 + e_5 + e_7 + e_9) - (f_1 + f_2 + f_3 + f_5) + g$
Sex x sight x rate x date	3	$c_3 - (d_2 + d_3 + d_6 + d_9) + (e_1 + e_3 + e_4 + e_6 + e_7 + e_{10}) - (f_1 + f_2 + f_4 + f_5) + g$
Sex x weight x rate x date	18	$c_4 - (d_4 + d_5 + d_6 + d_{10}) + (e_2 + e_3 + e_4 + e_8 + e_9 + e_{10}) - (f_1 + f_3 + f_4 + f_5) + g$
Sight x weight x rate x date	18	$c_5 - (d_7 + d_8 + d_9 + d_{10}) + (e_5 + e_6 + e_7 + e_8 + e_9 + e_{10}) - (f_2 + f_3 + f_4 + f_5) + g$

TABLE 3 (continued)

Source of Variation	d.f.	Sum of Squares
2nd order		
Sex x sight x weight	6	$d_1 - (e_1 + e_2 + e_5) + (f_1 + f_2 + f_3) - g$
Sex x sight x rate	3	$d_2 - (e_1 + e_3 + e_6) + (f_1 + f_2 + f_4) - g$
Sex x sight x date	1	$d_3 - (e_1 + e_4 + e_7) + (f_1 + f_2 + f_5) - g$
Sex x weight x rate	18	$d_4 - (e_2 + e_3 + e_8) + (f_1 + f_3 + f_4) - g$
Sex x weight x date	6	$d_5 - (e_2 + e_4 + e_9) + (f_1 + f_3 + f_5) - g$
Sex x rate x date	3	$d_6 - (e_3 + e_4 + e_{10}) + (f_1 + f_4 + f_5) - g$
Sight x weight x rate	18	$d_7 - (e_5 + e_6 + e_8) + (f_2 + f_3 + f_4) - g$
Sight x weight x date	6	$d_8 - (e_5 + e_7 + e_9) + (f_2 + f_3 + f_5) - g$
Sight x rate x date	3	$d_9 - (e_6 + e_7 + e_{10}) + (f_2 + f_4 + f_5) - g$
Weight x rate x date	18	$d_{10} - (e_8 + e_9 + e_{10}) + (f_3 + f_4 + f_5) - g$

TABLE 3 (continued)

Source of Variation	d.f.	Sum of Squares
1st order		
Sex x sight	1	$e_1 - (f_1 + f_2) + g$
Sex x weight	6	$e_2 - (f_1 + f_3) + g$
Sex x rate	3	$e_3 - (f_1 + f_4) + g$
Sex x date	1	$e_4 - (f_1 + f_5) + g$
Sight x weight	6	$e_5 - (f_2 + f_3) + g$
Sight x rate	3	$e_6 - (f_2 + f_4) + g$
Sight x date	1	$e_7 - (f_2 + f_5) + g$
Weight x rate	18	$e_8 - (f_3 + f_4) + g$
Weight x date	6	$e_9 - (f_3 + f_5) + g$
Rate x date	3	$e_{10} - (f_4 + f_5) + g$
Sex	1	$f_1 - g$
Sight	1	$f_2 - g$
Weight	6	$f_3 - g$
Rate	3	$f_4 - g$
Date	1	$f_5 - g$
TOTAL	447	$a - g$

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EDWARD K. STRONG, JR. *Vocational Interests of Men and Women*, Stanford University, California: Stanford University Press, 1943. Pp. xxix + 746. $6.50

A REVIEW

The *Strong Vocational Interest Blank* has made important history in the field of measurement. Since the appearance of the first form in 1927, the test has probably been the subject of more investigation and research than any other single test. And this research has been largely conducted or inspired by Strong himself.

Vocational Interests of Men and Women is a comprehensive review of problems in the field of the measurement of interests and a report of the prodigious researches of the writer with his *Vocational Interest Blank*, as well as of selected research by others bearing directly on his own. The emphasis is upon work since the publication of Fryer's notable *Measurement of Interests* in 1931. After an introductory survey of the field, Strong takes up in turn the questions of differentiation of occupations, interest factors, guidance based on interests, the differentiation of superior and inferior members of a group, the differentiation of skilled tradesmen, and the construction and scoring of an interest inventory.

The mass of evidence presented by Strong does not lend itself to summarization in as small a compass as a review. Suffice it to say that the field of the measurement of interests through use of the Strong *Vocational Interest Blank* is covered systematically, completely, and in an interesting style. The data are reported in an impartial vein; the con's as well as the pro's bearing upon the various procedures used by Strong are reported in fine good humor. The reader may not always agree with the conclusions reached by Strong, but in any case he must recognize that the book is a model of scientific reporting. No one can consider himself abreast of the interest field if he is not familiar with the contents of this book.

One fundamental point, perhaps, deserves particular mention. Thurstone's application of factor analysis to Strong's interest scales in 1931 pointed to the possibility of a new technique in obtaining scores for various occupations. Since Thurstone demonstrated that the variance of the occupational scores could be almost entirely accounted for by four factors, it followed logically that formulas could be developed by multiple correlation for obtaining occupational scores from measures of these four factors or from any set of a few selected measures which would account for the variance of all the scales. The actual scoring could then be reduced to the fundamental measures used. Later research by Strong, using more scales, increased the number of factors involved, but the implication of the findings is the same.

It has been a matter of surprise to some people in the field that Strong has not followed up this lead. Strong's reasons for not doing so, as given in his book, are not convincing. The first and less important reason is that "occupational scores can be obtained by machine scoring with no less trouble than with calculations involving multiple correlations." I am inclined to doubt this assertion. Strong is comparing a streamlined scoring method done on a mass production

basis with a straight computing machine job of calculating scores using regression weights. There is no reason why the application of a standard series of weights can not also be streamlined and done on a mass production basis.

Strong's other reason is by far the more important, since it concerns validity. Strong cites Dwyer's data to the effect that "the multiple correlation scores will correlate '.80 or better' in only the majority of cases with the results obtained by use of occupational scales." At this point we should stop to consider the purpose of the scales. If one particular scoring method is taken as the criterion, then we are doomed to labor in vain to develop a more valid method, for we can not hope to improve on what is implicitly assumed to be perfect to begin with. A more logical basis of judgment appears to be to determine which method is more successful in differentiating those in an occupation from men-in-general. It has not been demonstrated that such differentiation is accomplished better by scoring items than by assigning multiple regression weights* to a few selected measures which account for most of the interest variance in the occupations studied. Scores obtained by the latter procedure should be more valid than those obtained by the item-scoring method. Use of the item-scoring method carries no assurance that the optimal weight is given to each factor, since the number of items representing each factor is not controlled and item intercorrelations are not allowed for in determining item weights. Theoretically, it is impossible for an occupational scale obtained by Strong's method to differentiate better than weighted selected measures, if the measures used account for all the variance of the occupational scales. There is no reason to believe that the theoretical indication will not be borne out by the evidence, when obtained.

G. Frederic Kuder

* It should be noted that these regression weights must be developed through use of the original data from the various occupational groups, and *not* by using the occupational scale scores as criteria in developing the formulas.

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RULES FOR PREPARATION OF MANUSCRIPTS FOR PSYCHOMETRIKA JUNE 1944

1. Send manuscripts to
   HAROLD GULLIKSEN
   Managing Editor Psychometrika
   College Entrance Examination Board, Box 592
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2. If possible, submit three typewritten copies of the manuscript. Failure to adhere to this rule greatly delays our editorial procedure.
3. Use heavy white typewriter paper, size 8-1/2 x 11. Double-space the lines and allow wide margins for editorial work.
4. Tables should always be typed on separate sheets of paper, numbered, and referred to in the text by number, e.g., Table 2. If a table has a title, it should be typed at the head of the table, with initial letters capitalized. Footnotes referring to any part of a table should be placed immediately below the table. Necessary horizontal and/or vertical rulings should be indicated by author.
5. Footnotes should be reduced to a minimum. However, if the number of references does not exceed 8 to 10, they may be handled as footnotes, thus making unnecessary a list of references. Formulas in footnotes should be avoided. Footnotes should be indicated by the following sequence of symbols: * (asterisk or star), † (dagger), ‡ (double dagger), § (section mark), ‖ (parallels), ¶ (paragraph mark). All footnotes should be typed at the bottom of the page of text to which they refer.
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